DIRECTIONAL STABILITY OF AUTOMATICALLY STEERED BODIES.

By N. Minorsky.

1. INTRODUCTION.

The problem of directional stability of automatically steered ships is gradually becoming of increasingly greater importance for various reasons.

The possibility of obtaining more accurate steering by automatic means than can be accomplished by manual control with its inherent limitations due to the low sensitiveness of the human eye in detecting slow angular motions, fatigue, etc., becomes of greater importance with the increase in size of ships and cost of fuel.

For merchant ships an accurate and reliable automatic steering device becomes a real money saving proposition, largely justifying its use.

On battleships, by its use the absence or reduction of yawing in action means a better efficiency in gunfire, increased maneuvering speed and also a greater cruising radius.

In the case of airships, especially for long distance trips, directional stability is also of importance because the behavior of direction indicating instruments is then more satisfactory which leads to a still better stabilization, so that the certainty and safety of aeronavigation is thereby increased to an additionally greater degree.

A considerable number of other applications of the problem might be cited, such, for instance, as the radio control of ships and torpedos where the certainty and the accuracy of steered course must be assured between the moments when the radio signals establish a new setting of the controlling means.

Furthermore, quite recently, piloting vessels in harbors and channels by means of so-called radio cables carrying alternating current met with considerable success in this country and abroad*; it is likely that these installations will gradually replace older methods based on optical and acoustical signals. In that case an accurate and reliable automatic steering device enabling a vessel to follow very closely her "radio track" may be of considerable importance.

We may take a step further and predict some probable developments in the near future, which are already available, as far as principle is concerned and the existing handicaps are merely a matter of technical details.

Remarkable progress of recent years in connection with directive reception by coil antennas gives reason to believe that radiogoniometric methods of piloting ships or airships across still longer distances, such as trans-atlantic trips, before long will become an accomplished fact.

The main obstacle to it at the present time consists in the absence of a sufficiently powerful source of high frequency radiation adequate for radio compass performance.† There can be no doubt, however, that in the near future this obstacle will be overcome, especially in view of the continued progress along the line of high frequency generation by means of vacuum tubes.

The practical solution of the problem apparently will be reached when those technical difficulties are finally eliminated and when besides the magnetic and the true meridians given by magnetic or gyroscopic compasses respectively, the radio meridians or radio-routes will be established on the earth's surface.‡

* A. Crossley, JOURNAL OF A. S. N. E., Feb., 1921. L. A. Herdt and R. W. Owens, Comptes Rendus, Jan. 3, 1921.

† In this connection the work of Lt. Cr. A. Hoyt Taylor, U. S. N. R. F., can be consulted; Bureau of Standards No, 353; the same subject is treated in Comptes Rendus, Jan. 3, 1921.

‡ Such attempts have already been made in Germany during the war. See Jahrbuch der Drahtl. Telegr., Feb., 1920.

We may therefore state that even at the present time we have various means to ascertain a fixed direction on the surface of earth and to utilize it accordingly either for the purpose of manual steering, or generally speaking, for the automatic steering, if we connect the rudder with the direction indicating apparatus whatever its nature may be (magnetic, gyroscopic or radio compass) by any appropriate means, for example by means of an ordinary follow up system.

The question therefore arises whether such solution is correct? We know from past experience with automobile torpedoes that such solution is quite possible. Experience, however, shows that this simple method is sufficient only when the mass, or better the moment of inertia of the steered body, is not great and when very accurate steering is not the main object. The difficulties of such positional control increase very rapidly with the increased size of the vessel. It is also known that no good steersman will steer the ship by observing only the compass card without taking into consideration the state of angular velocity of the vessel; such a steering would be very inefficient in so far as the effect (action of the rudder) would come always too late after the cause (the original disturbance) which contributes to the permanent yawing.

An efficient helmsman keeps the ship accurately on her course by exerting a properly timed "meeting" and "easing" action on the rudder, i.e., by taking into consideration the elements characterizing the motion from the dynamical standpoint, namely, the instantaneous angular velocity of yawing as well as its time variations.

It has often been stated that the human intuition of the helmsman cannot be replaced by any mechanical contrivance whatever its nature may be. Such a standpoint seems to be erroneous, as far as the problem of automatic steering is concerned, since there is not so much question of intuition as of suitable timing based on actual observation. Once the element of observation is removed from the helmsman, there can be no accurate steering whatever his intuition may be.

The question, therefore, arises as to how the observation must be co-related to the timing of the rudder in order to obtain accurate steering?

If, therefore, accurate steering is nothing more than a special kind of timing of the rudder complicated by the inertia of the body to be steered, we may expect to be able to establish analytically what kind of timing must be adopted in order to reach the best possible conditions for directional stability of the body to be steered on its course.

An attempt to establish such fundamental relations for automatic steering is contained in the following.

2. LIMITATIONS UNDER WHICH THE PROBLEM CAN BE INVESTIGATED ANALYTICALLY.

The problem of automatic steering of moving bodies cannot be handled mathematically for the case of unlimited angular motion since there is no analytical expression applicable to the various torques acting on a ship, in general. It is practically necessary, therefore, to limit this investigation to the case where the mathematical solution has a real physical meaning. For this reason the stability of angular motion has been considered only for small deviations of the steered body from the desired direction which not only simplifies considerably the analytical solution of the problem but gives the only solution that is of practical interest, since only small deviations are possible, if the steering is to be accurate; if there is no stability for small deviations, it means that there is no stability in general since the ship will be continuously deviated from her course.*

* This method of attack is generally known as *method of small oscillations*. For more information see Webster, Dynamics, p. 157.

We may assume, therefore, that the angle of deviation of the ship from her course α, the angular velocity $\frac{d\alpha}{dt}$ and the angular acceleration $\frac{d^2\alpha}{dt^2}$ of yawing are very small, belonging to the first order of small quantities with which we are now concerned, and that the higher orders may be neglected.

The magnitude of the first order quantities may be fixed arbitrarily by assigning to them the small values that are determined by practical considerations.

For example, it may be assumed that the angle of yawing must not exceed $\frac{1}{4}°$ (.0043 radians), and that the angular velocity of yawing is 3.5×10^{-3} radians/sec., which values may be considered as limiting from the standpoint of their practical detection by the eye;* the square of those values being of the second order may be neglected. Having thus established the first order for the angles and velocities, the order of magnitude for angular accelerations can easily be ascertained, and the range within which the problem is studied is thus completely determined.

To fix ideas by a concrete example, the angular motion of a seagoing surface ship will be studied, although the method is general and can be applied to other cases.

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* The above figure 3.5×10^{-3} rad./sec. for angular velocity results from a series of observations carried out by the writer in 1916 on the sensitiveness of the eye in detecting angular velocities during a short interval of time (2–5 seconds). These observations were made by placing helmsmen on a revolving table, the angular velocity of which was known to the writer both in magnitude and direction, but was unknown to the observer; a very good line of bearing was available and absence of rolling and pitching made the observations very easy. Nevertheless, no certainty was observed in detecting either the rotation or its direction (being given 2–5 sec. for observation), when the table was rotating slower than $3.5 \times 10^{-3} - 4.5 \times 10^{-3}$ radians/sec., which corresponds to 30–40 minutes for one complete revolution of the table. It may be mentioned that during the time these tests were made a gyroscopic angular velocity indicator, designed by the writer, was found to detect the velocity corresponding to the vertical component of the earth's rotation (latitude 60°) i.e. a rotation about 50 times smaller in rate than the angular velocity of the table above referred to; the angular momentum of the gyroscope used in those tests was about 2.8×10^6 gr – cm.^{2} – sec.$^{-1}$, the gyroscopic precessional torque by means of which the earth's rotation was manifested being about 170 gr. cm.

Assuming that a ship is to steer a course α_0, the angular motion to be controlled by the rudder depends upon the single variable, or coordinate angle α giving the departure of the ship from her true course α_0, as well as upon its time derivatives.

The other coordinates, defining completely the angular position of the steered body at any instant, namely the angles of roll and pitch, as well as their time derivatives, are not considered here since they are not controllable by the rudder.

Since the conditions of directional stability do not depend upon any particular value of the course α_0 we may assume, for the sake of simplicity that $\alpha_0 = 0$ and consider only the deviations α from the true course due to yawing.

3. GENERAL FEATURES OF ANGULAR MOTION, VARIOUS TORQUES ACTING ON A SHIP.

In analyzing the turning motion of ships, Sir P. Watt says:

"The turning motion of a ship may be divided into three stages:

(*a*) When the rudder is first put over and the pressures on the hull are those necessary to produce angular acceleration.

(*b*) When accelerational forces are combined with those caused by the resistance of the ship to rotation, and

(*c*) When finally turning uniformly in a circular path.

The character of the forces acting during the states (*a*) and (*c*) can be ascertained and the type of motion under the complex conditions represented by (*b*) will consist of a gradual replacement of the motion at (*a*) by that at (*c*)."

It is apparent that for the purpose in view only the (*a*) stage of the motion is of interest.

We may enumerate now the torques acting on a ship and causing the initial turning motion as follows:

1. External disturbing torque due to any cause whatever (wind, waves, propellers, etc.)
2. Restoring torque due to rudder.
3. Resistance of the ship to turning.

There are no exact data relative to the nature and magnitude of the resistance of a ship to turning. Various authorities believe, however, that this resistance follows laws similar to those for rolling resistance, and that, probably, it may be expressed with practical accuracy by some law of proportionality to a certain power of angular velocity of yawing; there is no exact agreement, however, among investigators as to the exact value of the exponent, but it is generally agreed that it is somewhere between 1 and 2, varying according to a complex law. But, while the exact law, giving the resistance to turning as a function of angular velocity within a wide range of variations of angular velocity, remains unknown, we may safely assume that this law is linear within the narrow range of such variations around the zero velocity point.

Analytically this results from the well known method of approximation by infinitesimal analysis, according to which a small arc of continuous curve may be replaced by its chord without committing an error greater than the second order of small quantities, which therefore may be neglected.

The resisting torque, therefore, may be represented by:

$-\mathrm{B}\frac{d\alpha}{dt}$, where B is constant and the minus sign expresses the fact that the resisting torque always acts *against* the direction of angular velocity.

The other two torques acting on a ship are:

C (ρ) the restoring torque due to the action of the rudder, which is in general, a certain known function of the angle of rudder (ρ).

D, the disturbing torque deviating the ship from her course which may be due to any cause whatsoever as wind, waves, propellers, etc.; in general, this torque is so irregular that it cannot be taken into account analytically. It will be shown, however, that stability conditions do not require any mathematical expression for this torque.

Applying the principle of inertia in angular motion we have the expression A $\frac{d^2\alpha}{dt^2}$ equals the sum of all the combined external torques acting on the ship; A being the *effective* moment of inertia of the ship about the vertical axis passing through its center of gravity; it is generally greater than the moment of inertia of the ship on account of the inertia of water displaced by yawing, therefore,

$$A\frac{d^2\alpha}{dt^2} = -B\frac{d\alpha}{dt} - C\ (\rho) + D$$

The reason for the minus sign for $-B\frac{d\alpha}{dt}$ has been given above. It will be shown later that analogously C (ρ) must also have the minus sign.

We may, therefore, write:

$$A\frac{d^2\alpha}{dt^2} + B\frac{d\alpha}{dt} + C\ (\rho) = D$$

Since the function C (ρ) is always a continuous function of the angle ρ because an infinitely small increase in the restoring torque corresponds to an infinitely small increase of this angle and the angle ρ varies during accurate steering very little from its midship position ($\rho = o$), we may expand the function C (ρ) in Taylor's series around the point $\rho = o$ and thus write:

$$C\ (\rho) = C(o) + \rho\left(\frac{dC}{d\rho}\right)_{\rho=o} + \frac{\rho^2}{2'}\left(\frac{d^2C}{d\rho^2}\right)_{\rho=o} + \frac{\rho^3}{3'}\left(\frac{d^3C}{d\rho^3}\right)_{\rho=o} + \ldots\ (\ \ )$$

When the angle of rudder is zero obviously there is no restoring torque, thus C $(o) = o$. Further, ρ being a very small angle the terms containing higher powers of ρ may be neglected; we have then approximately

$$C(\rho) = \left(\frac{dC}{d\rho}\right)_{\rho=0} \times \rho$$

where

$$\left(\frac{dC}{d\rho}\right)_{\rho=0} = k$$

is a constant quantity characterizing the area of the rudder, lever arm of the rudder pressure, etc.

This gives a considerable simplification of the problem because under conditions that are of practical interest we may assume with sufficient accuracy that the restoring torque of the rudder is simply proportional to its angular position, the constant of proportionality k being equal to

$$\left(\frac{dC}{d\rho}\right)_{\rho=0},$$

that is, to the value of the derivative of the rudder torque in respect to the angle of rudder for $\rho = 0$.

The character of the function C (ρ) is generally known; thus according to Joessel*

$$C(\rho) = P \cdot L = 4.6 \text{ K A}' \text{ V}^2 \text{ L} \frac{\text{Sin } \rho}{G + H \text{ Sin } \rho},$$

where

P is the total pressure on the rudder in lbs.

V is the speed of ship in knots.

A′ is the area of the rudder in square feet.

K is a constant (for battleships and cruisers K varies between .6 and .75).

* Hibbard, Transactions A. I. E. E., 1914, p. 634.

L is lever arm of the rudder pressure equal to $l+d$, where l is the distance from the forward edge of the rudder to the center of gravity and d the distance of the center of pressure from the forward edge of the rudder.

G and H being two empirical constants; G = .39 and H = .61. This gives

$$k=\left(\frac{dC}{d\rho}\right)_{\rho=0}=\frac{4.6\ K\cdot A'\ V^2\ L}{G}.$$

4. THE PROBLEM OF AUTOMATIC STEERING.

The equation assumes then the form:

$$A\frac{d^2\alpha}{dt^2}+B\frac{d\alpha}{dt}+k\ \rho=D \qquad (2)$$

The problem is completely determined if the relation of the angle of rudder ρ as a function of α and its time-derivatives is given.

The equation (2) is the basis for the various methods of automatic steering.

In the following only those devices that give a continuous regulation to the rudder will be considered because only such devices are able to follow closely the development of angular motion of the ship which is always continuous with time. An adequate reaction on the part of the rudder in such a case continuously follows the action of disturbance and keeps it in check at all times.

It is also only in the case of a continuous regulation that the problem can be attacked by analysis.

Several methods of regulation are possible; we may mention about two fundamental classes according to whether the regulation affects the angle of rudder or the rate of change of this angle.

Theoretically one could go still further and imagine the third class of devices in which the controlling means would act upon angular acceleration of the rudder, but this is of no practical interest as will be shown later.

The above fundamental methods of rudder control can be expressed by simple linear relations of the form:

$$\rho = m\,\alpha + n\frac{d\alpha}{dt} + p\frac{d^2\alpha}{dt^2} \quad \text{(x)} \qquad \text{First class.}$$

$$\frac{d\rho}{dt} = m_1\alpha + n_1\frac{d\alpha}{dt} + p_1\frac{d^2\alpha}{dt^2} \quad \text{(xx)} \qquad \text{Second class.}$$

$$\frac{d^2\rho}{dt^2} = m_2\alpha + n_2\frac{d\alpha}{dt} + p_2\frac{d^2\alpha}{dt} \quad \text{(xxx)} \qquad \text{Third class and so on.}$$

For the sake of better understanding the individual features of each component control derived from various controlling means, several particular cases are treated in the beginning. Thus in studying the first case, we will first assume that $n = 0$, $p = 0$ which brings the fundamental relation of rudder control to a simple form $\rho = m \cdot \alpha$; then we will consider the other two sub-classes of the first class namely, the cases when $m = 0$, $p = 0$ and when $m = 0$, $n = 0$ in the above general relation (x) characterizing the first class of rudder control.

This study is contained in the following section 5 under the headings (*a*) (*b*) (*c*) and the general cases corresponding to the equations (x) (xx) are treated in sections 6 and 7. It is apparent that the number of possible combinations is very considerable; no attempt is made, therefore, to treat them all. The general method of attacking the problem shown in the following permits, however, to extend easily the analysis to any particular combination.

The practical method of combining the above mentioned controlling means can be made of course in several ways and it is not intended to go into consideration of them all; incidentally it may be mentioned that a mechanical differential gear affords perhaps one of the simplest solutions of such a linear combination; thus, for instance, if one component gear of the differential is turned over an angle proportional to $m\,\alpha$, and the other

$$n\,\frac{d\alpha}{dt}$$

the planetary wheel is turned over an angle proportional to

$$\frac{1}{2}\left(m\,\alpha+n\,\frac{d\alpha}{dt}\right).$$

and thus will afford the requisite linear combination. By adding through a second differential gear an angle proportional to

$$p\,\frac{d^2\alpha}{dt^2}$$

one will have on the planetary shaft of this second differential the complete linear relation

$$m\,\alpha+n\,\frac{d\alpha}{dt}+p\,\frac{d^2\alpha}{dt^2}$$

and can utilize it accordingly.

It is obvious that the above mentioned linear combination of controlling means is algebraic, i.e., holds not only for the magnitude of

$$m\,\alpha,\ n\,\frac{d\alpha}{dt},\ p\,\frac{d^2\alpha}{dt^2}$$

but also for their respective signs.

5. DEVICES OF THE FIRST CLASS (CONTROL OF THE ANGLE OF RUDDER).

As stated above, the controlling means act directly upon the angle of rudder; several methods of regulation are possible; they are successively treated in the following.

(a) Positional control of the angle of rudder.

This case is the simplest and oldest in the art; almost all inventions dealing with the problem of automatic steering are based on this principle which may be described as follows.

As soon as the ship deviates from her course over an angle a suitable mechanism moves the rudder over an angle ρ proportional to the angle of deviation (or yawing) α.

Analytically one may express this case by the very simple relation $\rho = \pm m\, \alpha$, where m is a coefficient of proportionality, the double sign meaning that, in general, there are two ways of connecting the azimuth control to the rudder.

Substituting $\rho = \pm m\, \alpha$ into the equation (2) and putting $km = C$ we have:

$$A \frac{d^2\alpha}{dt^2} + B \frac{d\alpha}{dt} \pm C\alpha = D \qquad (3)$$

This is the well known equation for the pendulum; the motion is given by the relation:

$$\alpha = Me^{x_1 t} + Ne^{x_2 t},$$

x_1 and x_2 being the roots of the auxiliary equation $Ax^2 + Bx \pm C = 0$ (3′) namely:

$$x = \frac{B}{2\,A} \pm \sqrt{\frac{B^2 - 4\,AC}{4\,A^2}}$$

It is known that in order to have stability of motion the auxiliary equation (3′) must have no positive real roots which means that only the plus sign must be used before C; from the practical standpoint the above analytical condition simply means that the positional (otherwise: azimuth) control must be connected to the rudder in such a way that whenever a deviation α occurs a mechanism controlled by that deviation must so act as to oppose its further increase which thus constitutes a correct connection.

For analogous reasons the minus sign signifies that the controlling mechanism is connected with the rudder in the wrong way.

Depending upon whether $B^2-4\,AC$ is positive or negative there will be either exponential decay of yawing or damped oscillatory motion.

Denoting–

$$\frac{B}{2\,A}$$

by u and the positive value of the root

$$\sqrt{\frac{B^2-4\,AC}{4\,A^2}}$$

by v, we have in the first case the solution of the form: $\alpha = Me^{-(u+v)t} + Ne^{-(u-v)t}$; and in the second the solution of the form: $Me^{-ut}\,\text{Cos}\,(vt+s)$, where M, N, s are constants of integration determined by boundary conditions. For a given ship B and A may be regarded as constant. C is a parameter characterizing the degree of the rudder action for a given value of deviation α; for example, the larger the area of the rudder, the greater will be the constant C; likewise, for the same rudder, the greater the angle ρ for a given angle of deviation α—the greater C will be. Thus for a gradually increasing value of C a "dead beat" steering may become oscillatory and vice versa.

If C is very small (i.e. a very weak rudder action) u is nearly equal to v so that $u-v=0$ which means that it will take a long time to bring the ship back to her course, once she has deviated therefrom by some reason; this is also obvious for the reason that it was assumed that the action of the rudder is small. By gradually increasing C the stability of the course will be increased but the steering becomes oscillatory or nearly so.

The damping out of oscillations caused by a disturbance depends upon

$$u = \frac{B}{2A};$$

for gradually increasing size of ships the moment of inertia increases apparently more rapidly than the friction to turning B. Sir Philip Watt considers that in the initial phase of yawing the value of B is very small.

This, probably, constitutes the main reason why positional control, although being much used in steering smaller craft, such as automobile torpedoes, and the like, has been found inadequate for steering larger vessels; this also can be understood from the fact that those devices are incapable of performing any "easing" or "meeting" action attributed generally to the helmsman's intellect.

From the above study it is apparent that the real cause for such a deficiency is in the fact that the conditions of damping out the yawing are not obtained by a purely positional rudder control.

(b) Angular velocity control of angle of rudder.

Another method of rudder control is obtained when the angle of rudder is varied in proportion to the value of instantaneous angular velocity of yawing recorded for example by a gyroscopic angular velocity indicator.* This case, consequently is characterized by a relation between the angle of rudder ρ and angular velocity, of the form:

$$\rho = \pm n \frac{d\alpha}{dt},$$

where n again stands for a suitable coefficient of proportionality expressing the degree of coupling between the angle of rudder and the angular velocity of yawing. In-

* Those devices called gyro-turn indicators or gyrometers are based on the principle of measuring instantaneous angular velocity by registering precessional reactions of a gyroscope constrained to move with the vessel. A device of that nature, compensated for the rolling and pitching disturbances, is described in U. S. Patent 1,372,184.

troducing this expression in the fundamental equation of automatic steering, and omitting the disturbing torque D* we have:

$$A\frac{d^2\alpha}{dt^2}+(B+E)\frac{d\alpha}{dt}=0$$

where $E=nk$ is the term relating to intensity of rudder action, depending upon area of the rudder, as well as upon the coupling or between the angle of rudder ρ and the angular velocity of the vessel.

The auxiliary equation corresponding to the differential equation of motion is of the form: $Ax^2+(B+E)x=0$ whose roots are:

$$x=0;\ x=-\frac{B+E}{A}.$$

The motion is consequently of the form:

$$\alpha=M+Ne^{-\frac{B+E}{A}t},$$

where M and N are again the integration constants; to determine them we may again assume certain boundary conditions. For example, let the motion be considered from the moment when a certain disturbance, having deviated the ship over an angle α_0, ceases to act. This gives for $t=0$

$$\alpha=\alpha_0;\ \frac{d\alpha}{dt}=\left(\frac{d\alpha}{dt}\right)_0$$

from which it follows that

$$\alpha_0=M+N;\ \alpha=\alpha_0+\left(\frac{d\alpha}{dt}\right)_0\frac{A}{B+E}\left(1-e^{-\frac{B+E}{A}t}\right) \qquad (4)$$

$$\left(\frac{d\alpha}{dt}\right)_0=-N\frac{B+E}{A};\ \frac{d\alpha}{dt}=\left(\frac{d\alpha}{dt}\right)_0 e^{-\frac{B+E}{A}t} \qquad (4')$$

* It is known that the intrinsic characteristics of an oscillating system are perfectly determined by the differential equation without the second member; the latter accounting merely for a particular solution depending upon the nature of an individual extraneous disturbance; for that reason the second member D may be disregarded in studying the oscillating features of the system. Some particular solutions corresponding to various second members are treated in the following.

Equations (4) (4′) show that both the deviation α and its angular velocity vary as exponential functions of time; the angle α gradually approaches a particular value

$$\alpha_{t=\infty}=\alpha_0+\left(\frac{d\alpha}{dt}\right)_0\frac{A}{B+E},$$

depending upon the initial angular velocity

$$\left(\frac{d\alpha}{dt}\right)_0$$

and A, B, E; the angular velocity thus gradually dies out. It is interesting to note that the action of angular velocity control denoted by the coefficient E, enters into the equations on equal terms with B.

The motion occurs in just the same way as if the water instead of opposing a resistance B to the angular motion of the ship, were a more resisting (viscous) fluid able to oppose a resistance B+E greater than B. Thus the angular velocity control of the character described accounts for an increased damping.

Steering by angular velocity alone is impossible because there is no directive force to stabilize the ship on a definite course; it has been shown that the ship deviates from her course without coming back to it; the apparent or effective resistance to turning is thus modified owing to the action of angular velocity control.

(*c*) *Angular acceleration control of angle of rudder.*

The third method is based upon the assumption that the angle of rudder is at all times proportional to the instantaneous value of angular acceleration, which can also be recorded by suitable means. Inserting the relation

$$\rho=p\frac{d^2\alpha}{dt^2}$$

into the general equation, we have

$$(A+F)\frac{d^2\alpha}{dt^2}+B\frac{d\alpha}{dt}=0,$$

where $F=pk$ is the term for intensity of the rudder action; the character of the motion is the same as in the second case, if we substitute $A+F$ for A and B for $B+E$.

The motion develops in just the same way as if, instead of the ship having moment of inertia A, its inertia were greater $(A+F)$, or smaller $(A-F)$, according to the way in which the accelerational control is acting on the rudder.

The remaining conclusions are the same as in the second case.

6. GENERAL CASE OF ANGLE OF RUDDER CONTROL.

Combining the three previous cases one has finally:

$$(A\pm F)\ \frac{d^2\alpha}{dt^2}+(B\pm E)\frac{d\alpha}{dt}\pm C\alpha=0 \qquad (6)$$

The double signs indicate the possibility of connecting the respective controlling elements in two different ways, of which only one is generally useful for steering. The general equation above contains all possible solutions for controlling the angle of rudder in proportion to the angle of yawing α and its time derivatives. The physical meaning of various controlling means may be classified, as follows:

(*a*) Compass Control C (otherwise: positional control of the angle of rudder) constitutes the directive couple for stabilizing the ship on her course. The timing is correct, if the directional angle of the rudder corresponds adequately to the deviation of the ship to check such deviation. The instability of control, or the wrong connection, occurs whenever the rudder, instead of checking the deviation, contributes to its further increase. Analytically instability occurs whenever one of the roots of the equation is positive; in the case of imaginary roots when the real part of the imaginary quantity is positive.

(*b*) Angular velocity control (gyrometer control) accounts either for an increase (plus sign before E) or for a decrease (minus sign before E) of the natural damping due to the resistance of the water. For steering purposes it is, obviously, advisable to have a strongly increased damping, in order to approach the dead beat steering conditions; consequently the plus sign before E characterizing the intensity of the gyrometer action, which is of interest.

(*c*) Angular acceleration control F accounts, in similar way, for an increase (plus sign), or for a decrease (minus sign), in the apparent moment of inertia of the ship, characterized by the term $A \pm F$. A ship with a greater moment of inertia is less liable to be disturbed, but on the other hand, once disturbed the handling of the ship is more difficult. It appears, therefore, that in order to be able to use the accelerational control most effectively it should be introduced with the plus sign, when the disturbance first begins to act, and the action of the control should be reversed (minus sign) when the disturbance disappears. Consequently, the accelerational control is useful only when there is, so to speak, automatic selective discrimination between the beginning and ending of incoming disturbances.

In summing up, it may be stated that by proper application of the individual regulations denoted by C, E, F, we may obtain any steering characteristic which is desirable in an individual case.

Thus, for example, in the case where a very accurate dead beat course is to be steered, predominance should be given to the angular velocity control for increasing the damping effect in steering. In case of a rough sea, where a certain freedom of steerage direction is even advisable to avoid too violent action of the rudder, angular velocity control may be decreased, while the compass control C may be given predominance and so on.

It is interesting to note that in the case where the angular velocity (or gyrometer) action is negative and equal to B, the total effective resistance (B − E) vanishes, and the angular motion of the ship becomes a sinusoidal undamped motion of a period depending upon the intensity of compass action C. Under such conditions the ship is able automatically to steer a sinusoidal, zig-zag course of any desired period.

The method above described for controlling the rudder is efficient from the standpoint of damping out the effect of a disappeared disturbance, but will not eliminate the effect of a steadily acting disturbing torque, such, for instance, as a steady wind. In order to steer the right course the ship must then have a certain amount of rudder consistent with the magnitude of the disturbing torque; on the other hand for no angular deviation, or no angular velocity, there is inevitably no angle of rudder, since the latter is supposed to be proportional to those quantities. It follows therefrom that the ship will be deviated from her course under the influence of such a steady disturbing torque D over an angle

$$\alpha_1 = \frac{D}{C},$$

representing analytically the particular solution of equation (3) corresponding to the constant second member D. Since the torque D may vary with the time, the angle α_1 is also variable; the steered course becomes indefinite under those conditions.

This first class of steering devices, acting to regulate the angle of rudder, is impractical for this reason.

The following class of steering devices has no such disadvantage, while all the advantages inherent to the first class of devices are present in the second class now to be described.

7. SECOND CLASS OF AUTOMATIC STEERING DEVICES (RATE OF MOVEMENT CONTROL OF THE RUDDER).

In this second class of steering devices the angle of yawing α and its time derivatives do not control *the angle* of rudder, but *the rate at which this angle is varied.* Since for small angles, the restoring torque due to the rudder is proportional to the rudder angle, *we may also characterize this second class as devices in which the controlling elements*

$$\alpha, \frac{d\alpha}{dt}, \frac{d^2\alpha}{dt^2}$$

act *upon the variations of the restoring torque due to the rudder and not upon the final value of the torque, as in the first case.* The second case, consequently, is characterized by a relation of the form

$$\frac{d\rho}{dt} = m\alpha + n\alpha' + p\alpha''.^*$$

Substituting this expression in the fundamental equation of automatic steering, we have:

$$A\frac{d^2\alpha}{dt^2} + B\frac{d\alpha}{dt} + k\int (m\alpha + n\alpha' + p\alpha'')\,dt = D.$$

Differentiating and rearranging the terms, we have:

$$A\frac{d^3\alpha}{dt^3} + (B + kp)\frac{d^2\alpha}{dt^2} + kn\frac{d\alpha}{dt} + km\alpha = \frac{dD}{dt}.$$

In the case of a steadily acting disturbance

$$\frac{dD}{dt} = 0,$$

from which follows the remarkable result that such a disturbance has no influence upon the performance of the

* Where m, n, p are coefficients for the degree of coupling between the speed of steering gear and angle, angular velocity, and angular acceleration of yawing, respectively.

device, depending solely upon the inertia A of the ship, the resistance B and constants, m, n, p, representing the intensities of the corresponding components of the control.

Such a device is responsive solely to the rate of increase or decrease in the action of the disturbance, and is not influenced by its actual magnitude; a device of this nature will steer the ship independently of any steady disturbances, and if the latter should vary, the steering will be adjusted automatically to the new value of the disturbance without affecting the correctness of the steered course.

The above result can also be understood from a more elementary discussion. Since in this case there is no rigid proportional relation between the elements of angular motion (α, α', α'') and the angle of rudder, the latter adjusts itself automatically until the motion is stopped; the ship is thus able to carry the required amount of rudder, being always on her right course.

The first member of the equation shows that the relations are totally different from those which were established in the case of the equation for devices belonging to the first class; namely,

1. The differential equation of motion is of the third order; in the first case it was of the second order.

2. None of the controlling means affect the angular inertia in this case; in the first case the accelerational control affects the inertia by the additional term F, whereas in this case it affects the resistance B by the term kp.

3. Velocity control by the gyrometer now appears as an independent term in the equation.

4. Only the positional control by the compass remains in the equation the same as in the first case.

A complete solution of the auxiliary equation of the third degree is not necessary for the purpose in view. The condition of stability of the ship on her course may be analytically expressed by the condition that none of the roots of the auxiliary equation may contain any positive real part. If all three roots are real, the condition of stability requires that they must all be negative. If one root is real and there is a pair of conjugate imaginary roots, the real root must be negative and the real part of imaginary roots must also be negative.

M. A. Blondel has shown* that by applying the so-called Hurwitz Theorem of Analysis, the "stability of roots" of any algebraic equation of the nth degree can easily be established in the form of a series of simultaneous conditions, which he calls Hurwitz conditions of stability. Referring for more detail on the subject to the article by Blondel, we may simply say that in the case of an equation of the third degree of the form $b_0x^3+b_1x^2+b_2x+b_3=0$, the stability conditions are:

$$(1)\ b_1>0$$
$$(2)\ b_1b_2-b_0b_3>0$$
$$(3)\ b_3\,(b_1b_2-b_0b_3)>0$$

In view of the second condition the last condition is merely $b_3>0$; so that finally we have (1) $b_1>0$; (2) $b_1b_2-b_0b_3>0$; (3) $b_3>0$. In the case under consideration the auxiliary equation is: $Ax^3+(B+kp)\,x^2+kn\,x+km=0$.

The Hurwitz conditions for stability of an automatically steered ship on her course become then:

$$(1)\ (B+kp)>0$$
$$(2)\ (B+kp)\,kn-A\,km>0$$
$$(3)\ km>0$$

The last condition means simply that the compass (more generally the direction indicating apparatus) must be connected in "the right way" to the rudder, i.e., when the ship yaws to port the helm must be moved to starboard.

The first condition shows that stability may be upset if the accelerational control kp is connected in the wrong direction so as to render $B+kp<0$.

* Journal de Physique, April and May, 1919.

The second condition completely determines the stability of the ship on her course; it shows that for a sufficiently great value of n, representing the intensity of gyrometer control, the stable conditions can always be obtained, whatever the inertia of the vessel may be.

It is interesting to note that the gyrometer action does not *add itself* to the resistance of the medium B, as in devices of the first class, but is *multiplied by it;* in other words, in the case of absence of velocity control ($n = 0$), stability disappears because the second condition becomes $-A\,km$ which is always negative because both A and km are essentially positive.

From the above relations we may conclude that accelerational control is of no special importance, and can be dispensed with; while velocity control (gyrometer) on the contrary constitutes the necessary and sufficient condition for stability of the steered course.

8. GENERALIZATION.

It has been shown that the two methods of rudder control are equally good, so long as there is no permanent disturbing torque acting on the ship. On the other hand, the conditions are totally different, if such a disturbance is acting; the "follow up" control of the rudder is then unable to maintain the ship on her course, while "the rate of angle control" is independent of any steady disturbances, but the ship is steered correctly provided the disturbance itself remains steady; if the disturbance is variable, it is only this variation that figures as the effective disturbance, and affects the instantaneous behavior of the device, but not its absolute value.

We may, therefore, proceed still further, and establish a third class of automatic steering devices in which the elements of angular motion of the ship control the acceleration of the rudder, and not its angle as in the first case, nor its velocity as in the second case. Such a condition is expressed by a relation of the form

$$\frac{d^2\rho}{dt^2} = m_2\alpha + n_2\alpha' + p_2\alpha'',$$

which being substituted in the general equation for automatic steering, finally results in the differential equation

$$A\frac{d^4\alpha}{dt^4} + B\frac{d^3\alpha}{dt^3} + kp_2\frac{d^2\alpha}{dt^2} + kn_2\frac{d\alpha}{dt} + km_2\alpha = \frac{d^2D}{dt^2}.$$

Such a device will not be influenced either by permanent steady disturbances or by uniformly varying disturbances. It is doubtful, however, whether such a method is necessary or advisable for practical purposes in view of the character of the disturbances usually occurring. In general three different types of disturbing torques have to be taken into account: (1) steadily acting, (2) periodical, (3) accidental or irregular.

The first kind of disturbances is practically the only one that is of interest for big ships because they generally respond, not to individual disturbing impulse, but to the time integral of many impulses due to the waves whose action is thus cumulative with time; to the same category of disturbances belongs the wind, the strength of which does not, in general, vary abruptly; to the same kind of disturbing factors belong also the influence of trim, depending upon distribution of the cargo and similar permanent causes.

The second kind of disturbances is important only for smaller craft, responding instantaneously to each individual impulse or wave (which may become of importance for larger ships in a heavy sea); this disturbance constitutes what is generally called yawing among waves; in this case the periodicity of such a forced yawing is obviously the apparent periodicity of the waves, and the rudder can do nothing to prevent it; in practice, in such cases intentionally loose steering is generally admissible in order not to over-regulate the rudder too much, which would only uselessly decrease the speed; this is also the only possible method in the case of automatic steering.

The third kind of disturbances is quite special and hardly likely to arise on seagoing ships, but occurs more frequently in aerial navigation (air pockets, bumps); the only way to overcome the effect of such disturbances rapidly is to diminish the time lag of the controlling transmissions to a minimum.

In view of the foregoing it is problematical whether the third class of rudder control can become of any practical interest. The first class of automatic steering devices (based on follow up steering gear) is decidedly objectionable in view of uncertainty of the steered course under the action of disturbances, although when no such disturbances are acting it yields quite a correct solution of the problem.

The devices of the second class seem to be the logical solution of the problem of automatic steering.

9. INFLUENCE OF TIME LAG OF TRANSMISSION UPON THE AUTOMATIC STEERING.

In the foregoing analysis it has been assumed that the restoring action of the rudder is instantaneously responsive to the elements of angular motion

$$\alpha, \frac{d\alpha}{dt}, \frac{d^2\alpha}{dt^2}$$

controlling the rudder. In reality it takes an appreciable time before such action begins; the delay is due to the transmission system possessing a certain amount of inertia, either mechanical or electromagnetic, lost motion, viscosity and similar causes. By perfecting the transmitting means the lag may be reduced, but never totally eliminated. Any transmission system can thus be characterized in terms of its lag measured, for example, in seconds.

In other words, while the real motion at a certain moment t is characterized by the angle $\alpha(t)$, angular vleocity $\alpha'(t)$, angular acceleration $\alpha''(t)$, the instantaneous rudder torque, existing at the same moment of time t, relates to the moments of past time, lagging by the intervals T_1, T_2, T_3 behind the compass, the gyrometer, and the acceleration-transmitting systems, respectively.

Considering only the second class of steering devices, the differential equation of angular motion becomes then:

$$A\,\alpha''(t)+B\,\alpha'(t)+k\int\Big[m\alpha(t-T_1)+ n\alpha'(t-T_2)+p\alpha''(t-T_3)\Big]dt=D$$

Differentiating, we have:

$$A\,\alpha'''(t)+B\,\alpha''(t)+km\,\alpha(t-T_1)+ kn\,\alpha'(t-T_2)+kp\,\alpha''(t-T_3)=\frac{dD}{dt}$$

The functional character of α, α', α'' etc. in respect to time is emphasized by the symbols $\alpha(t)$, $\alpha'(t)$, $\alpha''(t)$.

Since the lag in time may be made very small by suitable design and careful construction, the intervals of time T_1, T_2, T_3 are very small quantities in comparison with the period of yawing of the ship. This fact permits the use of well known approximations resulting from the expansion of Taylor's Theorem, and we thus have:

$$\alpha(t-T_1)=\alpha(t)-T_1\,\alpha'(t)+\frac{T_1^2}{2'}\,\alpha''(t)-\frac{T_1^3}{3'}\,\alpha'''(t)+$$

$$\alpha'(t-T_2)=\alpha'(t)-T_2\,\alpha''(t)+$$

$$\frac{T_2^2}{2'.} \alpha''' (t) - \frac{T_2^3}{3'.} \alpha'''' (t) +$$

$$\alpha'' (t - T_3) = \alpha'' (t) - T_3 \alpha''' (t) +$$

$$\frac{T_3^2}{2'.} \alpha'''' (t) - \frac{T_3^3}{3'.} \alpha''''' (t) +$$

Neglecting the small quantities of the second and higher orders, we have:

$$\alpha (t - T_1) = \alpha (t) - T_1 \alpha' (t) = \alpha - T_1 \alpha'$$
$$\alpha' (t - T_2) = \alpha' (t) - T_2 \alpha'' (t) = \alpha' - T_2 \alpha''$$
$$\alpha'' (t - T_3) = \alpha'' (t) - T_3 \alpha''' (t) = \alpha'' - T_3 \alpha'''$$

substituting these values for $\alpha (t - T_1)$ etc. in the equation we have:

$$(A - kpT_3) \alpha''' + (B + kp - knT_2) \alpha'' + k (n - mT_1) \alpha' + km \alpha = \frac{dD}{dt}$$

and the Hurwitz-Blondel conditions of stability become:

(1) $B + kp - knT_2 > 0$

(2) $(B + kp - knT_2) k (n - mT_1) - (A - kpT_3) km > 0$

(3) $km > 0$

It will be seen, therefrom, that the conditions of stability are affected by delay in transmission, except in the case of the last condition, which remains unchanged.

The first condition shows that for a sufficiently great value of lag T_2 in gyrometer transmission the stability may be upset; for gradually increasing lag T_2 this will happen the sooner, the smaller the resistance of the medium to the angular motion of the ship.

The second condition shows that the time lag T_1 in compass transmission (more generally in transmission controlled by a direction indicating apparatus) diminishes the useful effect of the gyrometer control.

It is to be noted that the lag T_3 in accelerational control (if there be any) may even be beneficial, and contribute to stability of the steered course, by decreasing the effective angular inertia of the ship to a certain extent.

CONCLUSION.

From the preceding study it is apparent that all possible methods of rudder control *do not actually anticipate* the disturbing angular motion, but merely utilize this motion at its beginning when its value is small for the purpose of impressing a properly timed reaction against it. It is therefore obvious that the disturbing angular motion must necessarily occur *before* any controlling means can be operative.

Starting from rest under the influence of an external disturbance, all elements of the angular motion (angle, angular velocity, angular acceleration and still higher time derivatives) appear simultaneously and at the limit $t=0$ it is impossible to answer the question "which is better for the control": angular acceleration, angular velocity or angle. Outside of this isolated point this indefiniteness disappears and the problem has a quite definite physical meaning insofar as each higher time-derivative passes through its characteristic stages (maxima, zero points, etc.) at an earlier epoch than a corresponding lower one.

Thus for example the angular velocity being analytically the time integral of angular acceleration it must necessarily depend upon the past values of acceleration which account for its present magnitude, in other words, in the physical succession of facts the angular acceleration always precedes the angular velocity and the latter precedes the angle of deviation. It is therefore of a decided advantage for the purpose of a sufficiently rapid control to utilize preferably higher time derivatives of angular motion.

But while the problem of anticipating the appearance of a disturbance theoretically is meaningless—it may have a practical meaning if we agree to understand that by it considerable reduction of yawing in comparison with the manual control, now in use, can be effected.

The progress in this direction is clearly indicated and may be summarized as follows:

(*a*) Perfection of instruments detecting the angular motion ahead of the human eye.

(*b*) Perfection of methods utilizing these instruments for the purpose of a suitable timing consistent with the dynamical nature of angular motion.

The above paper must be considered as an attempt to solve the last mentioned problem.