A TREATISE

ON

THE STABILITY OF MOTION.

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ON THE

STABILITY OF A GIVEN STATE OF MOTION,

PARTICULARLY STEADY MOTION.

BEING THE ESSAY TO WHICH THE ADAMS PRIZE WAS ADJUDGED IN 1877, IN THE UNIVERSITY OF CAMBRIDGE.

BY

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PREFACE.

In March, 1875, the usual biennial notice was issued, giving the subjects for the Adams Prize to be adjudged in 1877. The following is the chief portion of the notice:

The University having accepted a Fund raised by several members of St John's College for the purpose of founding a Prize to be called the *Adams Prize*, for the best essay on some subject of Pure Mathematics, Astronomy or other branch of Natural Philosophy, the Prize to be given once in two years, and to be open to the competition of all persons who have at any time been admitted to a degree in this University—

The Examiners give notice that the following is the subject of the Prize to be adjudged in 1877: *The Criterion of Dynamical Stability*.

To illustrate the meaning of the question imagine a particle to slide down inside a smooth inclined cylinder along the lowest generating line, or to slide down outside along the highest generating line. In the former case a slight derangement of the motion would merely cause the particle to oscillate about the generating line, while in the latter case the particle would depart from the generating line altogether. The motion in the former case would be, in the sense of the question, stable, in the latter unstable.

The criterion of the stability of the equilibrium of a system is, that its potential energy should be a minimum; what is desired is, a corresponding condition enabling us to decide when a dynamically possible motion of a system is such, that if slightly deranged the motion shall continue to be only slightly departed from.

The essays must be sent in to the Vice-Chancellor on or before the 16th December 1876, &c., &c.

S. G. PHEAR, *Vice-Chancellor.*
J. CHALLIS.
G. G. STOKES.
J. CLERK MAXWELL.

The pressure of other engagements for some time prevented me from giving my attention to the subject. This essay was therefore almost entirely composed during the year 1876. It is now printed as it was sent in to the Examiners, the changes being merely verbal. Some few additions have been made where explanation appeared to be necessary, but all these have been marked by square brackets, so that they can be at once distinguished from the original parts of the essay.

In order to shorten the essay as much as possible many *merely algebraic* processes have been omitted and the results only are stated. It is hoped that this will add clearness as well as brevity to the reasoning, as the attention of the reader will not be called from the argument to follow a manipulation of symbols which may not present any novelty.

The line of argument taken may be indicated in a general way as follows. Chapter I. begins with some definitions of the terms *stable* and *steady* motions. It is then pointed out that whether the forces which act on the system admit of a force-function or not, the stability of the motion, if steady, is indicated by the nature of the roots of a certain determinantal equation. The boundary between stability and instability being generally indicated by the presence of equal roots, a criterion is investigated to determine beforehand whether equal roots do or do not imply instability. This case being disposed of, the consideration of the determinantal equation is resumed. Two general methods are given by which, without solving the equation, it may be ascertained whether the character of the roots imply stability or instability. These occupy Chapters II. and III. In the first method a derived equation is made use of, and it is shown that a simple inspection of the signs of the coefficients of the several powers in these two equations will decide the question of stability. In the second method a certain easy process is found which if performed on the determinantal equation will lead to the criteria of stability. At the end of the third Chapter a geometrical interpretation is given to the argument.

In the fourth Chapter the forces which act on the system are supposed to have a force-function. The determinantal equation is then much simplified. Several points are considered in this Chapter which are necessary to the argument, such as the proper method of choosing the steady co-ordinates (if there be any), the distinction between harmonic oscillation about steady motion and that about equilibrium, and the changes which must be made in the determinantal equation when the equations of Lagrange become inapplicable. A method of modifying the Lagrangian function is also given by which, in certain cases, the fundamental determinant may be reduced to one of fewer rows and columns.

In the fifth Chapter a series of subsidiary determinants is formed, and it is shown that at least as many of the conditions of stability are satisfied as there are variations of signs lost in the series in passing from one given state to another. It is also shown that this is equivalent to a maximum condition of the Lagrangian function.

In the sixth Chapter the energy test of stability is considered. It is also shown that, when the motion is steady, this reduces to the same criterion as that indicated in Chap. V.

In the seventh Chapter the question considered is whether the stability of a state of motion can really be determined by an examination of the terms of the first order only. In some cases these are certainly sufficient, and an attempt is made to discriminate between these cases and those in which the terms of the higher order ultimately alter the character of the motion.

If the Hamiltonian characteristic and principal functions be given, the conditions of stability as regards space only, or both space and time may be deduced. But if these be not known as expressed in the Hamiltonian form, we may yet sometimes distinguish between stability and instability if we can determine whether a certain integral ceases to be a minimum at some instant of the motion. This is the subject of the eighth chapter.

As part of the third edition of my treatise on the Dynamics of Rigid Bodies was written at the same time as this essay, there are necessarily points of contact between the two works. Thus the subjects of the first part of the seventh chapter and of a portion of the sixth will be found discussed in the treatise on Dynamics. But as the objects of the two books are not the same, it will be found that in all these cases there are considerable differences in the modes of demonstration.

EDWARD J. ROUTH.

PETERHOUSE,
August 14, 1877.

CONTENTS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

CHAPTER I.

Definitions of the terms small quantity, stable motion, steady motion. *Arts.* 1, 2.

A system of bodies in steady motion is stable if the roots of a certain determinantal equation are such that their real parts are all negative. Art. 3.

Effect of equal roots, and a test to determine whether equal roots do or do not introduce terms which contain the time as a factor. Arts. 4—7.

Object of Chapters II. and III. Art. 8.

1. Let us suppose a dynamical system to be set in motion under any forces and to move in some known manner. If any small disturbance be given to the system, it may deviate only slightly from its known motion, or it may diverge further and further from it. Let θ, ϕ, &c. be the independent variables or co-ordinates which determine the position of the system, and let the known motion be given by $\theta = \theta_0$, $\phi = \phi_0$, &c. where θ_0, ϕ_0, &c. are known functions of the time t. To discover the disturbance of the system we put $\theta = \theta_0 + x$, $\phi = \phi_0 + y$, &c. These quantities x, y, &c. are in the first instance very small because the disturbance is small. The quantities x, y, z, &c. are said to be *small* when it is possible to choose some quantity numerically greater than all of them, which is such that its square can be neglected. This quantity may be called the standard of reference for small quantities.

If, after the disturbance, the co-ordinates x, y, z, &c. remain always small, the undisturbed motion is said to be *stable;* if, on the other hand, any one of the co-ordinates become large, the motion is called *unstable.*

It is clear that the same motion may be stable for one kind of disturbance and unstable for another. But it is usual to suppose the disturbance *general,* so that if the motion can be made unstable by any kind of disturbance (provided it be small) it is said to be unstable. On the other hand, it will be called stable only when it is stable for *all* kinds of small disturbances.

2. To determine whether x, y, z, &c. remain small, we must substitute for θ, ϕ, &c. in the equations of motion their values $\theta_0 + x, \phi_0 + y$, &c. Assuming that x, y, &c. remain small, we may neglect their squares, and thus the resulting equations will be linear in x, y, z, &c. The coefficients of $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$, &c. in these equations may be either constants or functions of the time. In the former case the undisturbed motion is said to be *steady* for these co-ordinates, in the latter *unsteady*. In the case of a steady motion x, y, z, &c. are all functions of the time which has elapsed since the disturbance and of certain constants of integration which are determined by the initial values of $x, \frac{dx}{dt}, y, \frac{dy}{dt}$, &c. We may therefore define a steady motion to be such that the same change of motion follows from the same initial disturbance at whatever instant the disturbance is communicated to the system.

If all the coefficients in the equations to find x, y, z are constant, they may be made to contain t by a change of co-ordinates. Thus we may write for x, y, z, &c.

$$x = \alpha\xi + \beta\eta + \dots$$
$$y = \alpha'\xi + \beta'\eta + \dots$$
$$z = \text{\&c.}$$

where α, β, &c. are any functions of t we please. Conversely, when the coefficients are functions of t, we may sometimes make the coefficients constant by a proper change of co-ordinates. But this cannot always be done. If there are n co-ordinates, we have n^2 arbitrary functions α, β, &c. at our disposal. In each of the n linear equations of motion we may have three terms for each co-ordinate, and thus we have $(3n-1)n$ coefficients to make constants. We have therefore in general too many equations to satisfy. The proper method of choosing the co-ordinates of reference will be considered in a future chapter.

3. Let us suppose a dynamical system to be making small oscillations under the action of any forces which may, or may not, possess a force function and to be subject to any resistances which vary as the velocities of the parts resisted. The general equations of motion will then be of the form

$$\left.\begin{aligned} \left(A_2\frac{d^2}{dt^2} + A_1\frac{d}{dt} + A_0\right)x + \left(B_2\frac{d^2}{dt^2} + B_1\frac{d}{dt} + B_0\right)y + \text{\&c.} = 0 \\ \left(A_2'\frac{d^2}{dt^2} + A_1'\frac{d}{dt} + A_0'\right)x + \left(B_2'\frac{d^2}{dt^2} + B_1'\frac{d}{dt} + B_0'\right)y + \text{\&c.} = 0 \\ \text{\&c.} = 0 \end{aligned}\right\}.$$

To solve these equations we write

$$x = Me^{mt}, \quad y = M'e^{mt}, \text{ \&c.}$$

Substituting we obtain a determinantal equation to find m. If we put

$$A = A_2m^2 + A_1m + A_0, \qquad B = B_2m^2 + B_1m + B_0,$$
$$A' = A_2'm^2 + A_1'm + A_0', \qquad \text{\&c.}$$

this equation may be written in the simple form

$$\begin{vmatrix} A, & B, & C\ldots\ldots \\ A', & B', & C'\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = 0.$$

We may also write the equation in the form $f(m) = 0$.

The coefficients M, M', &c. are not independent, but if we represent the minors of A, B, C, &c. by a, b, c, &c. we may easily show that

$$\frac{M}{a} = \frac{M'}{b} = \frac{M''}{c} = \text{\&c.}$$

We also have

$$\frac{M}{a'} = \frac{M'}{b'} = \frac{M''}{c'} = \text{\&c.}$$

It may be shown by properties of determinants that these equations all give the same ratios. If Δ_2 be the second minor obtained from the determinant $f(m)$ by omitting the first and second rows and columns, we know that

$$\Delta_2 f(m) = ab' - a'b.$$

Hence if $f(m) = 0$ we have $\dfrac{a}{a'} = \dfrac{b}{b'}$.

In the same way we may show that $\dfrac{a}{a'} = \dfrac{c}{c'}$, and so on. This property of Determinants is given in Dr Salmon's *Higher Algebra*, Lesson IV. Ex. 1.

The general solution of the equation may therefore be written in the form

$$\left.\begin{aligned} x &= L_1a_1e^{m_1t} + L_2a_2e^{m_2t} + \ldots \\ y &= L_1b_1e^{m_1t} + L_2b_2e^{m_2t} + \ldots \\ z &= \text{\&c.} \end{aligned}\right\},$$

where L_1, L_2 ... are arbitrary constants, a_1, a_2, &c. the values of the minor a when m_1, m_2 ... are substituted for m; b_1, b_2 ... the values of the minor b when similar substitutions are made, and so on.

4. We see that the whole character of the motion will depend on the signs of the quantities $m_1, m_2 \dots$ If any one be real and positive, x, y, &c. or some of them will ultimately become large, and the steady motion about which the system is oscillating will be unstable. If all the roots are real, negative or zero and unequal, the motion will be stable.

If two of the roots be imaginary we have a pair of imaginary exponentials. If these imaginary roots be $\alpha \pm \beta\sqrt{-1}$, the terms can be rationalized into

$$e^{\alpha t}(N_1 \cos \beta t + N_2 \sin \beta t).$$

The motion will be stable if α be negative or zero, and unstable if α be positive.

If two roots be equal, the form of the solution is changed. Let $m_2 = m_1 + h$ where h will be ultimately zero, we then have

$$x = L_1 a_1 e^{m_1 t} + L_2 \left(a_1 e^{m_1 t} + \frac{da_1}{dm} h e^{m_1 t} + a_1 h t e^{m_1 t}\right).$$

If we now make L_1 and L_2 infinite in the usual manner, we find

$$x = \left\{M_2 a_1 t + M_2 \frac{da_1}{dm} + M_1 a_1\right\} e^{m_1 t},$$

$$y = \left\{M_2 b_1 t + M_2 \frac{db_1}{dm} + M_1 b_1\right\} e^{m_1 t},$$

$$\&c. = \&c.,$$

where M_1, M_2 are two arbitrary constants which replace L_1, L_2.

In the same way if three roots are equal we have

$$x = \left[M_3\left(a_1 \frac{t^2}{2} + \frac{da_1}{dm_1} t + \tfrac{1}{2} \frac{d^2 a_1}{dm_1^2}\right) + M_2\left(a_1 t + \frac{da_1}{dm_1}\right) + M_1 a_1\right] e^{m_1 t},$$

$$y = \left[M_3\left(b_1 \frac{t^2}{2} + \frac{db_1}{dm_1} t + \tfrac{1}{2} \frac{d^2 b_1}{dm_1^2}\right) + M_2\left(b_1 t + \frac{db_1}{dm_1}\right) + M_1 b_1\right] e^{m_1 t}.$$

This rule will be found convenient in practice to supply the defect in the number of arbitrary constants produced by equal roots. At present we are only concerned with their effect on the stability of the system. The terms which contain t as a factor will at first increase with t, but if m be negative, the term $t^n e^{mt}$ can never be numerically greater than $\left|\frac{n}{em}\right|^n$. If m be very small the initial increase of the terms may make the values of x and y become large, and the motion cannot be regarded as a small oscillation. But if the system be not so much disturbed that $M . \overline{\frac{n}{em}}\Big|^n$ is large, the terms will ultimately disappear and the motion may be regarded as stable. If, however, the real parts of the equal roots are positive or zero, the terms will become large and the motion will be unstable.

5. In some cases, however, the relations which exist between the coefficients are such that the terms which contain t as a factor are all zero. It is of some importance to discriminate these cases, for the stability of the system is then unaffected by the presence of equal roots.

Let us suppose first that the determinantal equation has two roots only equal to m_1, and let the terms depending on these be

$$x = (N_1 + N_2 t)\, e^{m_1 t},$$
$$y = (N_1' + N_2' t)\, e^{m_1 t},$$
$$\&c. = \&c.$$

Substituting in the equations of Art. (3) we have, following the same notation as before,

$$\left.\begin{aligned} AN_2 + BN_2' + CN_2'' + \ldots &= 0 \\ A'N_2 + B'N_2' + C'N_2'' + \ldots &= 0 \\ \&c. &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots \text{I.}$$

$$\left.\begin{aligned} AN_1 + BN_1' + \ldots &= -\frac{dA}{dm} N_2 - \frac{dB}{dm} N_2' - \ldots \\ A'N_1 + B'N_1' + \ldots &= -\frac{dA'}{dm} N_2 - \frac{dB'}{dm} N_2' - \ldots \end{aligned}\right\} \ldots\ldots \text{II.}$$
$$\&c. = \&c.$$

To avoid entering more minutely than is necessary into the properties of linear equations, we shall assume that these equations for the given value of m lead to but one solution with two of the N's arbitrary, unless the determinantal equation has more than two roots equal to m_1. If in this unique solution the N_2's are all zero we must have

$$\left.\begin{aligned} AN_1 + BN_1' + \ldots &= 0 \\ A'N_1 + B'N_1' + \ldots &= 0 \\ \&c. &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{III.}$$

Since two of the constants N_1, N_1', &c. are to be arbitrary, let them be N_1, N_1', then since $\frac{N_1}{a} = \frac{N_1'}{b}$ we must have the minors a and b each equal to zero. Also since $\frac{N_1}{a} = \frac{N_1''}{c}$, we shall have N_1'' infinite unless $c = 0$. In the same way we may prove that all the other first minors are zero.

And if the first minors are zero, we may show that two of the equations may be deduced from the others. Let the symbol $\begin{bmatrix} A\,B \\ A'B' \end{bmatrix}$ represent the second minor, with the usual sign, formed by omitting the rows and columns in which $AB\,A'B'$ occur.

Then since the minors a, b, c, &c. are zero, we have

$$\left.\begin{aligned} A'\begin{bmatrix} AB \\ A'B' \end{bmatrix} + A''\begin{bmatrix} AB \\ A''B'' \end{bmatrix} + \ldots &= 0 \\ B'\begin{bmatrix} AB \\ A'B' \end{bmatrix} + B''\begin{bmatrix} AB \\ A''B'' \end{bmatrix} + \ldots &= 0 \\ \&c. &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{IV.}$$

Omitting the first line of III. let us multiply the others by $\begin{bmatrix} AB \\ A'B' \end{bmatrix}\begin{bmatrix} AB \\ A''B'' \end{bmatrix}$ &c., respectively. Adding the results, we have an identity. Hence the second equation may be deduced from the others which follow it. In the same way, the first equation may be deduced from the others.

Rejecting the first two equations, let us transpose the arbitrary constants N_1 and N_1' to the right-hand sides of the remaining equations. If there are to be only two arbitrary constants, these remaining equations must be independent; solving, we have

$$\begin{bmatrix} AB \\ A'B' \end{bmatrix} N_1'' = -N_1\begin{bmatrix} BC \\ B'C' \end{bmatrix} + N_1'\begin{bmatrix} AC \\ A'C' \end{bmatrix},$$

with similar equations for the others. Hence the constants N_1, N_1', &c., are connected by equations of the form

$$N_1\begin{bmatrix} BC \\ B'C' \end{bmatrix} - N_1'\begin{bmatrix} AC \\ A'C' \end{bmatrix} + N_1''\begin{bmatrix} AB \\ A'B' \end{bmatrix} = 0,$$

so that when any two are chosen as the arbitrary ones, the others may be deduced from them.

If the determinantal equation has three roots equal to m_1, and if the terms which contain t as a factor are all zero, the equations III. must admit of a solution with three of the constants

N_1, N_1', &c. arbitrary. If these be N_1, N_1', N_1'', we see that the second minors $\begin{bmatrix} BC \\ B'C' \end{bmatrix} \begin{bmatrix} AC \\ A'C' \end{bmatrix} \begin{bmatrix} AB \\ A'B' \end{bmatrix}$ must be zero.

Next since

$$N_1 \begin{bmatrix} BD \\ B'D' \end{bmatrix} - N_1' \begin{bmatrix} AD \\ A'D' \end{bmatrix} + N_1''' \begin{bmatrix} AB \\ A'B' \end{bmatrix} = 0,$$

we must have $\begin{bmatrix} BD \\ B'D' \end{bmatrix}$ and $\begin{bmatrix} AD \\ A'D' \end{bmatrix}$ both zero or N_1''' infinite.

Hence all the second minors are zero. And if the second minors are all zero, we have

$$A'' \begin{bmatrix} ABC \\ A'B'C' \\ A''B''C'' \end{bmatrix} + A''' \begin{bmatrix} ABC \\ A'B'C' \\ A'''B'''C''' \end{bmatrix} + \dots = 0,$$

and by similar reasoning three equations may be deduced from the remaining ones. We have then

$$N_1 \begin{bmatrix} BCD \\ B'C'D' \\ B''C''D'' \end{bmatrix} - N_1' \begin{bmatrix} ACD \\ A'C'D' \\ A''C''D'' \end{bmatrix} + N_1'' \begin{bmatrix} ABD \\ A'B'D' \\ A''B''D'' \end{bmatrix} - N_1''' \begin{bmatrix} ABC \\ A'B'C' \\ A''B''C'' \end{bmatrix} = 0,$$

with similar equations.

Since $f(m)$ is the determinant formed by eliminating N_1, N_1', &c. from III. we have

$$\frac{df(m)}{dm} = \frac{df(m)}{dA}\frac{dA}{dm} + \frac{df(m)}{dB}\frac{dB}{dm} + \text{\&c.} + \frac{df(m)}{dA'}\frac{dA'}{dm} + \text{\&c.}$$

$$= a\frac{dA}{dm} + b\frac{dB}{dm} + \text{\&c.} + a'\frac{dA'}{dm} + \text{\&c.}$$

This vanishes when a, b, &c., a', &c. are all zero. If therefore the first minors of $f(m)$ all vanish when $m = m_1$, the equation $f(m) = 0$ has two roots equal to m_1. In the same way $\frac{da}{dm}$ vanishes if all its first minors are zero. But

$$\frac{d^2f(m)}{dm^2} = a\frac{d^2A}{dm^2} + \frac{da}{dm}\frac{dA}{dm} + \text{\&c.}$$

vanishes if a, $\frac{da}{dm}$, &c. are all zero. If therefore the first and second minors of $f(m)$ all vanish when $m = m_1$, the equation

$f(m) = 0$ has three roots equal to m_1. It is evident the proposition may be extended to any number of roots.

The test that, when equal roots occur in the determinantal equation, the terms in the values of x, y, &c. *which contain* t *as a factor should be absent may be stated thus. If there are two equal roots all the first minors must vanish. If three equal roots, all the first and second minors must vanish, and so on. In these cases the equal roots introduce merely a corresponding indeterminateness into the coefficients.*

When there are more equal roots than there are rows in the determinantal equation, it is easy to see that there must be some terms in the integrals which contain t as a factor.

[The following simple example will illustrate the application of this test.

A particle is in equilibrium at the origin of co-ordinates under the action of forces whose force function U is given by

$$U = \tfrac{1}{2} Ax^2 + \tfrac{1}{2} By^2 + \tfrac{1}{2} Cz^2 + Dyz + Ezx + Fxy.$$

If the level surfaces are ellipsoids and the force acts inwards, it is clear that the equilibrium of the particle must always be stable. If then any equal roots occur in the determinantal equation, the test should show that the terms which contain t as a factor are absent.

If T be the semi vis viva of the particle and if its mass be taken as unity, we have

$$T = \tfrac{1}{2} x'^2 + \tfrac{1}{2} y'^2 + \tfrac{1}{2} z'^2.$$

Omitting accents and forming the discriminant of $-m^2T + U$ we have the following determinantal equation:

$$\begin{vmatrix} A - m^2 & F & E \\ F & B - m^2 & D \\ E & D & C - m^2 \end{vmatrix} = 0.$$

This is the "discriminating cubic" which determines the axes of the quadric $U = c$, where c is a constant. The conditions that two of its roots should be equal, i.e. that the quadric should be a spheroid, are well known to be

$$A - \frac{EF}{D} = B - \frac{FD}{E} = C - \frac{DE}{F} = m_1^2,$$

where m_1^2 is equal to either root. These are just the conditions obtained by equating any first minor of the determinant to zero.

The conditions that three of the roots of the cubic should be equal, i.e. that the quadric should be a sphere, are

$$A = B = C,\ \ D = 0,\ \ E = 0,\ \ F = 0.$$

These are the conditions that every second minor should vanish.

In this example we have taken the case of a single particle. Similar remarks however apply when any system of bodies is disturbed from a state of stable equilibrium. The oscillations may be found by the method of Lagrange. The final determinantal equation may be conveniently formed by equating to zero the discriminant of $-m^2T + U$, where T is the semi vis viva with the accents denoting differentiations with regard to the time omitted, and U is the force function. [It is a known theorem that the existence of finite equal roots does not affect the stability of the equilibrium.] Hence the conditions for equal roots must be such as to make all the minors equal to zero. Conversely, this theorem will often conveniently give the conditions that Lagrange's determinant has equal roots.]

6. That there should be a difference in the modes in which equal roots affect the motion is no more than we should expect *a priori*. Suppose the coefficients of the equation $f(m) = 0$ to be functions of some quantity n, and that as n passes through the value n_0, two roots become equal to each other. Let the quadratic factor containing these roots be $m^2 + 2\alpha m + \beta$, and let us consider only the case in which α and β are real. We have $\alpha^2 - \beta = 0$ when $n = n_0$. If $\alpha^2 - \beta$ change sign as n passes through the value n_0, the roots will change from a trigonometrical to a purely exponential form, which would indicate a change from oscillatory to non-oscillatory motion. The passage from one kind of motion to the other may be effected through a motion represented by expressions having the time as a factor. But if $\alpha^2 - \beta$ does not change sign, for example, if it be a perfect square for all values of n, there will be no change from one kind of motion to the other, and in this case we should expect that the motion when the roots are equal will be represented by terms of the same character as before. Briefly, we may expect equal roots to introduce terms with t as a factor at the boundary between stability and instability; and to introduce merely an indeterminateness into the coefficients when the motion is stable on both sides.

It is easy to show that in the first of these two cases the minors could not contain either of the factors of $m^2 + 2\alpha m + \beta$. For since $\alpha^2 - \beta$ changes sign, these factors are in one case imaginary; and therefore if one factor occur in any minor the other must also be present. The minors would not only vanish, but must have equal roots also. But as in Art. (3), $\Delta_2 f(m) = ab' - a'b$.

Hence if all the first minors have equal roots it is clear that either $f(m)$ has more than two equal roots, or all the second minors must vanish. The latter is impossible unless $f(m)$ has more than two equal roots.

These general considerations are not meant to replace the proofs given in the last article, but merely to explain how a difference in the effects of the equal roots might arise.

7. Summing up what precedes, we see that if a dynamical system have n co-ordinates its stability depends on the nature of the roots of a certain equation of the $2n$th degree.

If the roots of this equation are all unequal, the motion will be stable if the real roots and the real parts of the imaginary roots are all negative or zero, and unstable if any one is positive. If several roots are equal the motion will be stable if the real parts of those roots are negative and not very small, and unstable if they are negative and small, zero, or any positive quantity. But if, as often happens in dynamical problems, the terms which contain t as a factor are absent from the solution, the condition of stability is that the real roots and the real parts of the imaginary roots of the subsidiary equation should be negative or zero.

8. When the equation $f(D) = 0$ is of low dimensions we may solve it or otherwise determine the nature of its roots; the stability or instability of the system will then become known. But if the degree of the equation be considerable this is not a very easy problem. We shall devote the two next Chapters to the consideration of two methods by either of which, without solving the equation, we can determine the conditions that the real roots and the real parts of the imaginary roots should be all negative. The determination of these conditions has, it appears, never before been accomplished.* The consideration of the equations of motion will then be resumed, and the form of the determinantal equation $f(D) = 0$ when the forces admit of a force function will be more particularly investigated.

* [These conditions for the cases of a biquadratic and a quintic had been found by the Author in 1873, and read before the London Mathematical Society in June, 1874. See also the third edition of the Author's *Rigid Dynamics*, Art. 436.]

CHAPTER II.

Statement of the theory by which the necessary and sufficient tests of stability are found. Objections to this theory. *Arts.* 1—3.

These tests shown to be integral functions of the coefficients. *Art.* 4.

Method of finding these tests when the coefficients of the equation are numerical, or when several terms are absent. *Arts.* 5, 6.

All these tests shown to be derivable from one called the fundamental term. *Arts.* 7, 8.

The fundamental term found as an eliminant. *Art.* 9.

A method of finding the fundamental term by derivation from the fundamental term of one degree lower. *Art.* 10.

Another and better method of doing the same by means of a differential equation. *Arts.* 11—13.

1. The object of this Chapter has been explained at the end of Chapter I. Briefly, the criterion that the motion of a system of bodies should be stable is that the roots of a certain equation should have all their real parts negative. We propose to investigate these conditions.

Let the equation to be considered be

$$f(x) = p_0 x^n + p_1 x^{n-1} + \dots + p_{n-1} x + p_n = 0.$$

Let the real roots be a_1, a_2 ... and the imaginary roots be

$$\alpha_1 \pm \beta_1 \sqrt{-1}, \quad \alpha_2 \pm \beta_2 \sqrt{-1}, \text{ \&c.}$$

Then

$$f(x) = p_0 (x - a_1)(x - a_2) \dots (x^2 - 2\alpha_1 x + \alpha_1^2 + \beta_1^2), \text{ \&c.}$$

If then a_1, a_2, &c. α_1, α_2, &c. are all negative, every term in each factor, and therefore in the product, must be positive.

It is therefore necessary that every term in the equation $f(x) = 0$ should have the same sign. It will be convenient to suppose this sign to be positive.

It is also clear on the same suppositions that none of the coefficients $p_0, p_1, \dots p_n$ can be zero, except when the roots of the equation are all of the form $\pm\beta\sqrt{-1}$, or when some of the roots are zero.

2. Let us now form the equation whose roots are the sums of the roots of $f(x)$ taken two and two. Let this be

$$F(x) = P_0 x^m + P_1 x^{m-1} + \dots + P_{m-1} x + P_m = 0,$$

where $m = n\dfrac{n-1}{2}$. The real roots of this equation will be $a_1 + a_2$, $a_1 + a_3$, &c. $2\alpha_1$, $2\alpha_2$, &c. and the imaginary roots will be $a_1 + \alpha_1 \pm \beta_1\sqrt{-1}$, &c. It is clear from the same reasoning as before that if a_1, a_2, &c. α_1, α_2, &c. are all negative, the coefficients P_0, P_1, &c. must all have the same sign.

Conversely, if p_0, $p_1 \dots$ have all the same sign, the equation $f(x)$ can have no real positive root, and if $P_0, P_1 \dots P_m$ have all the same sign the equation $F(x)$ can have no positive root, and therefore $f(x)$ can have no imaginary root with its real part positive.

3. *Our first test of the stability of a dynamical system is that all the coefficients of the dynamical equation* f (D) = 0 *and all the coefficients of its derived equation* F (D) = 0 *should have the same sign.*

It should be noticed that though these conditions are all necessary and sufficient, they are not all independent. We obtain too many conditions. In many cases, however, we can at once reduce them to the proper number of independent conditions, and when this is difficult we can have recourse to the second method, to be given in the next Chapter, which is free from this objection.

In order to apply this method with success, it is necessary to have some convenient methods of calculating the coefficients $P_0, P_1 \dots P_m$.

4. The first method which suggests itself is one similar to that usually given to determine the coefficients of the equation whose roots are the squares of the differences of the roots of any given equation.

If S_1, $S_2 \dots$ be the sums of the first, second powers, &c. of the roots of the equation $fx = 0$, we have by Newton's theorem

$$S_n + p_1 S_{n-1} + p_2 S_{n-2} + \dots = 0,$$

where p_0 has been put equal to unity. If Σ_1, $\Sigma_2 \dots$ be the sums of the powers of the equation $F(x) = 0$, we have in the same way

$$\Sigma_n + P_1\Sigma_{n-1} + P_2\Sigma_{n-2} + \dots = 0;$$

we may also prove

$$\Sigma_1 = (n-1) S_1,$$
$$\Sigma_2 = (n-2) S_2 + S_1^2,$$
$$\Sigma_3 = (n-4) S_3 + 3S_1S_2,$$
$$\Sigma_4 = (n-8) S_4 + 4S_1S_3 + 3S_2^2,$$
$$\Sigma_5 = (n-16) S_5 + 5S_1S_4 + 10S_2S_3;$$

and the general relation can be found without difficulty.

In this way we find

$$P_1 = (n-1)p_1,$$

$$P_2 = \frac{(n-1)(n-2)}{1.2} p_1^2 + (n-2)p_2,$$

$$P_3 = \frac{(n-1)(n-2)(n-3)}{1.2.3} p_1^3 + (n-2)^2 p_1p_2 + (n-4)p_3,$$

$$P_4 = \frac{(n-1)(n-2)(n-3)(n-4)}{1.2.3.4} p_1^4 + \frac{(n-2)^2(n-3)}{1.2} p_1^2p_2$$
$$+ (n-3)^2 p_1p_3 + \frac{(n-2)(n-3)}{1.2} p_2^2 + (n-8)p_4.$$

But the process becomes longer and longer at every stage. We shall therefore proceed to point out some other methods of obtaining the coefficients.

This method of proceeding has indeed been stated only because it proves in a convenient way that when $p_0 = 1$, all the coefficients $P_1, P_2, P_3 \ldots$ of the derived equation are integral rational functions of the coefficients $p_0, p_1 \ldots p_n$.

5. *The equation* f (x) = 0 *being given, to calculate the coefficients of* F (x) = 0.

Put $x = y \pm z$ and equate separately to zero the sums of the even and odd powers of z, we have

$$\left.\begin{aligned} f(y) + f''(y)\frac{z^2}{\lfloor 2} + f^{\text{iv}}(y)\frac{z^4}{\lfloor 4} + \ldots &= 0 \\ f'(y)z + f'''(y)\frac{z^3}{\lfloor 3} + \ldots &= 0 \end{aligned}\right\}.$$

Rejecting the root $z = 0$, let us eliminate z. Then the roots of the resulting equation in y are the arithmetic means of the roots of $f(x) = 0$.

If, on the other hand, we eliminate y we have an equation of an even degree to find z. This, putting $4z^2=\zeta$, is the equation whose roots are the squares of the differences of the roots of the given equation.

It may be thought that this elimination may prove tedious, but it will be presently shown that only the first and last terms of the result are really wanted. All the others may be omitted in the process of elimination, and thus the labour will be greatly lessened. The method is however most useful when the given equation has several of its terms absent.

6. *Example. To determine the condition that the roots of the biquadratic*

$$x^4+px^3+qx^2+rx+s=0$$

should indicate a stable motion.

Applying the rule we have

$$F(x)=x^6+3px^5+(3p^2+2q)x^4+(4pq+p^3)x^3$$
$$+(2p^2q+pr+q^2-4s)x^2+(pq^2+p^2r-4ps)x+pqr-r^2-p^2s.$$

The first four coefficients contain only positive terms, and need not be considered. If the last three coefficients be called P_4, P_5, P_6, we have

$$p^2P_4-4P_6=(pq-2r)^2+2p^4q+p^3r,$$
$$pP_5-4P_6=(pq-2r)^2+p^3r.$$

If then P_6 is positive, all the other coefficients are positive.

The necessary and sufficient conditions of stability are therefore that p, q, r, s should be finite and positive, and

$$P_6=pqr-r^2-p^2s$$

positive or zero.

7. In forming the derived equation $F(x)$ the only difficulty is to form the last term P_m. *For when this is known the other terms can be at once derived from it by an easy process.*

Let $a, b, c\ldots$ be the roots of $f(x)=0$ with their signs changed, and let

$$f(x)=p_0x^n+p_1x^{n-1}+\ldots+p_n.$$

Let Δ stand for the operation

$$\Delta=\frac{d}{da}+\frac{d}{db}+\frac{d}{dc}+\ldots$$

Then since $\frac{p_n}{p_0} = abc \ldots$ we have obviously

$$\Delta \frac{p_n}{p_0} = \frac{p_{n-1}}{p_0}.$$

In the same way we have

$$\Delta \frac{p_{n-1}}{p_0} = 2\frac{p_{n-2}}{p_0}.$$

And generally

$$\Delta \frac{p_{n-\kappa+1}}{p_0} = \kappa \frac{p_{n-\kappa}}{p_0},$$

and so on up to

$$\Delta \frac{p_1}{p_0} = n.$$

Let us now operate with Δ on any expression

$$\phi(p_0, p_1 \cdots p_n),$$

which has the same number of factors in every term. Let r be the number of factors, then ϕ may be written

$$\phi = p_0^r \phi_1 \left(\frac{p_1}{p_0}, \ldots \frac{p_n}{p_0}\right),$$

$$\therefore\ \Delta\phi = p_0^{r-1}\left\{np_0 \frac{d}{dp_1} + (n-1)p_1 \frac{d}{dp_2} + \ldots\right\} \phi_1$$

$$= \left\{np_0 \frac{d}{dp_1} + (n-1)p_1 \frac{d}{dp_2} + \ldots + p_{n-1}\frac{d}{dp_n}\right\} \phi.$$

Let P_0, $P_1 \ldots P_m$ be the coefficients of the derived equation $F(x)$, and let $P_0 = 1$. Then since the roots of $F(x)$ are the sums of the roots of $f(x)$ taken two and two, it is easy to see that

$$P_{m-1} = \tfrac{1}{2}\Delta P_m,$$
$$2P_{m-2} = \tfrac{1}{2}\Delta P_{m-1},$$
$$3P_{m-3} = \tfrac{1}{2}\Delta P_{m-2},$$
$$\&c. = \&c.$$

Thus when P_m is known, the other terms may be calculated without difficulty. The term P_m will be called the *fundamental term* of the equation.

Example. Given in the case of a biquadratic

$$P_6 = p_1p_2p_3 - p_0p_3^2 - p_1^2p_4,$$

to calculate P_5.

Performing the operation

$$p_3 \frac{d}{dp_4} + 2p_2 \frac{d}{dp_3} + 3p_1 \frac{d}{dp_2} + 4p_0 \frac{d}{dp_1},$$

on P_6 we find after division by 2,

$$P_5 = p_1 p_2^2 + p_1^2 p_3 - 4p_0 p_1 p_4,$$

which is the result already given.

8. It should be noticed that in the equation

$$f(x) = p_0 x^n + p_1 x^{n-1} + \dots + p_n = 0,$$

if we regard x as a number, $p_0 p_1 \dots p_n$ are all of equal dimensions. It follows from the theory of dimensions, that if any subject of operation be the sum of a number of terms of the form

$$p_0^\alpha p_1^\beta p_2^\gamma, \dots$$

there must be the same number of factors in every term. For example, in every term of the expression for P_m we have $\alpha + \beta + \gamma$ &c. the same.

On the other hand, we may regard x as a quantity of one dimension, and in this case $p_0, p_1 \dots p_n$ have their dimensions indicated by their suffixes. We must therefore have $\beta + 2\gamma + 3\delta + \dots$ as well as $\alpha + \beta + \gamma + \dots$ the same in every term.

These two tests of the correctness of our processes will be found convenient.

9. *The whole derived equation being known when the fundamental term is known, it is required to find the fundamental term.*

[First Method.]

If we write $-x$ for x in any equation, we have a second equation whose roots are equal and opposite to those of the first equation. If we eliminate x between these two, we shall get a result which must be zero when the two equations have a common root. The eliminant must therefore contain as a factor the product of the sums of the roots of the given equation taken two and two.

It will afterwards be shown that the last term P_m of the derived equation (when p_0 is put equal to unity) always contains the term

$$p_1, p_2, p_3 \dots$$

with a coefficient which is positive and equal to unity.

Hence we have this rule, to find P_m, eliminate x between

$$\left.\begin{array}{ll} x^n + p_2x^{n-2} + p_4x^{n-4} + \ldots & = 0 \\ p_1x^{n-1} + p_3x^{n-3} + \ldots & = 0 \end{array}\right\},$$

and divide the result by the coefficient of $p_1p_2p_3\ldots p_{n-1}$.

It is obvious that the result may be written down as a determinant. On trial, however, it will be found more convenient to make the elimination by the method of eliminating the highest and lowest terms than to expand the determinant.

10. *Given the fundamental term of the equation derived from*

$$f(x) = p_0x^{n-1} + p_1x^{n-2} + \ldots + p_{n-1} = 0 \ldots\ldots\ldots\ldots(1),$$

to find the fundamental term of the equation derived from

$$p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n = 0 \quad \ldots\ldots\ldots\ldots(2).$$

[Second Method.]

Let Q_n be the product of the sums, two and two, of the roots of the equation (2) taken *with their signs changed*, so that Q_n is the same as the fundamental term of the derived equation and differs from P_m only in having a suffix more convenient for our present purpose.

Let Q_n be expanded in a series of powers of p_n: thus

$$Q_n = \phi_0 + \phi_1p_n + \phi_2p_n^2 + \ldots$$

where ϕ_0, ϕ_1, ϕ_2, &c., are all functions of p_0, p_1, &c., which functions have to be found. Let the roots of $f(x) = 0$ with their signs changed be $a, b, c\ldots$, then

$$Q_{n-1} = (a+b)(a+c)\ldots$$

Let us introduce a new root, which, when its sign is changed, we shall call r. Then

$$Q_n = (a+b)(a+c)\ldots(r+a)(r+b)\ldots$$
$$= \frac{Q_{n-1}}{p_0}(p_0r^{n-1} + p_1r^{n-2} + \ldots + p_{n-1}).$$

This value of Q_n must be the same as that given by the series when we write

$$p_0, \quad p_1 + rp_0, \quad p_2 + rp_1, \quad \&c., \quad p_{n-1} + rp_{n-2}, \quad rp_{n-1},$$

respectively, for p_0, p_1, p_2, &c., p_{n-1}, p_n. Equating the coefficients of the terms independent of r we have

$$\phi_0 = Q_{n-1}\frac{p_{n-1}}{p_0}.$$

Equating the terms containing the first power of r we have

$$p_{n-1}\phi_1 + \left(p_0\frac{d}{dp_1} + p_1\frac{d}{dp_2} + \dots + p_{n-2}\frac{d}{dp_{n-1}}\right)\phi_0 = Q_{n-1}\frac{p_{n-2}}{p_0}.$$

Substituting for ϕ_0 we have, by a known theorem in the Differential Calculus,

$$\phi_1 = -\left(p_0\frac{d}{dp_1} + p_1\frac{d}{dp_2} + \dots + p_{n-2}\frac{d}{dp_{n-1}}\right)\frac{Q_{n-1}}{p_0}.$$

Equating the terms containing the second power of r we have

$$\frac{1}{2}\left\{p_0^2\frac{d^2}{dp_1^2} + 2p_0p_1\frac{d^2}{dp_1dp_2} + \dots\right\}\phi_0$$

$$+\left(p_0\frac{d}{dp_1} + p_1\frac{d}{dp_2} + \dots\right)\phi_1 \cdot p_{n-1}$$

$$+\phi_2 p^2_{n-1} = Q_{n-1}\frac{p_{n-3}}{p_0},$$

and so on. Thus we have

$$Q_n = Q_{n-1}\frac{p_{n-1}}{p_0} - p_n\left(p_0\frac{d}{dp_1} + p_1\frac{d}{dp_2} + \dots + p_{n-2}\frac{d}{dp_{n-1}}\right)\frac{Q_{n-1}}{p_0} + \&c.$$

If we examine this process, we see that when Q_{n-1} is known, we may at once write down the terms independent of p_n and the coefficient of p_n. The process to find the coefficient of p_n^2 is longer, but it may be much shortened by the consideration that when $p_0 = 1$ the result must be an integral function of the coefficients. We may therefore omit all terms as soon as they make their appearance, which do not contain the factor p^2_{n-1}, for we know that such terms must disappear from the result.

This method is not so convenient as the one which will be presently given to find the coefficients of the higher powers of p_n. But it is useful as showing that Q_n contains the term

$$\frac{p_1p_2\cdots p_{n-1}}{p_0^{n-1}},$$

with a positive integral coefficient equal to unity. This will be clear from the consideration that the term independent of p_n in Q_n is obtained from Q_{n-1} by multiplying by $\frac{p_{n-1}}{p_0}$. If therefore Q_{n-1} contains $\frac{p_1p_2\cdots p_{n-2}}{p_0^{n-2}}$, Q_n must contain the term $\frac{p_1p_2\cdots p_{n-1}}{p_0^{n-1}}$. No other term can be formed which is equal to this with an opposite sign, for the terms which enter by the other processes to be performed on Q_{n-1} all contain p_n as a factor. Now $Q_2 = \frac{p_1}{p_0}$; therefore Q_3 contains the term $\frac{p_1 p_2}{p_0^2}$, Q contains $\frac{p_1 p_2 p_3}{p_0^3}$, and so on.

It has been shown that every term in $p_0^{n-1} Q_n$ has the same number of factors (Art. 8). It follows from this reasoning that this number of factors is $n-1$.

11. *To find the fundamental term of the derived equation by means of a differential equation.*

[Third Method.]

The fundamental term required is a factor of the eliminant of

$$\left.\begin{array}{r} p_0 x^n + p_2 x^{n-2} + \ldots = 0 \\ p_1 x^{n-1} + p_3 x^{n-3} + \ldots = 0 \end{array}\right\}.$$

Let $x^2 = y$, then we have

$$\left.\begin{array}{l} p_0 y^{\frac{n}{2}} + p_2 y^{\frac{n}{2}-1} + p_4 y^{\frac{n}{2}-2} + \ldots = 0 \\ p_1 y^{\frac{n}{2}-1} + p_3 y^{\frac{n}{2}-2} + \ldots\ldots\ldots\ldots = 0 \end{array}\right\} n \text{ even},$$

$$\left.\begin{array}{l} p_0 y^{\frac{n-1}{2}} + p_2 y^{\frac{n-3}{2}} + \ldots\ldots\ldots\ldots\ldots = 0 \\ p_1 y^{\frac{n-1}{2}} + p_3 y^{\frac{n-3}{2}} + \ldots\ldots\ldots\ldots\ldots = 0 \end{array}\right\} n \text{ odd}.$$

If we write $y + dy$ for y the result of the elimination must be the same. Hence if we make

$$\left.\begin{array}{ll} \frac{dp_2}{dy} = \frac{n}{2} p_0, & \frac{dp_4}{dy} = \left(\frac{n}{2} - 1\right) p_2, \text{ \&c.} \\ \frac{dp_3}{dy} = \left(\frac{n}{2} - 1\right) p_1, & \frac{dp_5}{dy} = \left(\frac{n}{2} - 2\right) p_3, \text{ \&c.} \end{array}\right\} n \text{ even},$$

$$\left.\begin{array}{ll} \frac{dp_2}{dy} = \frac{n-1}{2} p_0, & \frac{dp_4}{dy} = \frac{n-3}{2} p_2, \text{ \&c.} \\ \frac{dp_3}{dy} = \frac{n-1}{2} p_1, & \frac{dp_5}{dy} = \frac{n-3}{2} p_3, \text{ \&c.} \end{array}\right\} n \text{ odd},$$

and if E be the eliminant, we have

$$\frac{dE}{dy} = 0.$$

It follows that whether n be even or odd, E must satisfy the equation

$$\left.\begin{aligned} & p_{n-2}\frac{dE}{dp_n}+p_{n-3}\frac{dE}{dp_{n-1}} \\ & +2\left(p_{n-4}\frac{dE}{dp_{n-2}}+p_{n-5}\frac{dE}{dp_{n-3}}\right) \\ & +3\left(p_{n-6}\frac{dE}{dp_{n-4}}+p_{n-7}\frac{dE}{dp_{n-5}}\right) \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}=0.$$

We may make the elimination by multiplying the two equations by y, $y^2\ldots$, until we have as many equations as we have powers to eliminate. If in the determinant thus formed, we multiply out the terms in the diagonal joining the right-hand top corner to the left bottom corner, we get when n is even $p_n^{\frac{n}{2}-1}\,p_1^{\frac{n}{2}+1}$ and when n is odd $p_n^{\frac{n-1}{2}}\,p_0^{\frac{n-1}{2}}$. Now Q_n must contain $n-1$ factors and be of the $n\frac{n-1}{2}$th degree. Hence when n is even $E=cp_1Q_n$ and when n is odd $E=cQ_n$, where c is some constant.

Now $\frac{d}{dp_1}$ does not occur in the above differential equation. Hence treating p_1 as a constant, we see that Q_n must satisfy the differential equation

$$\left(p_{n-2}\frac{dQ_n}{dp_n}+p_{n-3}\frac{dQ_n}{dp_{n-1}}\right)+2\left(p_{n-4}\frac{dQ_n}{dp_{n-2}}+p_{n-5}\frac{dQ_n}{dp_{n-3}}\right)$$
$$+3\left(p_{n-6}\frac{dQ_n}{dp_{n-4}}+p_{n-7}\frac{dQ_n}{dp_{n-5}}\right)+\&c.=0.$$

12. We may show that $p_0^{n-1}Q_n$ is a symmetrical function of the coefficients p_0, $p_1\ldots$, p_n and the same coefficients read backwards. Let a, b, $c\ldots$ be the roots of $f(x)=0$ with their signs changed, then $Q_n=(a+b)\,(a+c)\ldots.$

If now we read the coefficients in the opposite order, the roots of the equation thus formed will, when their signs are changed, be $\frac{1}{a}$, $\frac{1}{b}\ldots$. If Q'_n be the fundamental term of the equation derived from this, we have

$$Q'_n=\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{c}\right)\ldots.$$

Since $\frac{p_n}{p_0}=abc\ldots$ we see that

$$p_0^{n-1}Q_n=p_n^{n-1}Q'_n.$$

We may therefore infer that $p_0^{n-1}Q_n$ also satisfies the differential equation

$$\left(p_2\frac{dE}{dp_0}+p_3\frac{dE}{dp_1}\right)+2\left(p_4\frac{dE}{dp_2}+p_5\frac{dE}{dp_3}\right)+3\left(p_6\frac{dE}{dp_4}+p_7\frac{dE}{dp_5}\right)+\&c.=0.$$

13. *We may use either of these differential equations to find Q_n when Q_{n-1} is given.*

Let the first differential equation be represented by

$$\nabla Q_n = 0,$$

and let

$$Q_n = A_0 + A_1p_n + A_2p_n^2 + \ldots,$$

where A_0, A_1, ... are functions of p_0, p_1, &c. The value of A_0 has been proved in Art. 10 to be

$$A_0 = Q_{n-1}\frac{p_{n-1}}{p_0}.$$

To find the other coefficients of the powers of p_n substitute this value of Q_n in the differential equation; we have

$$\begin{aligned}0 = \nabla A_0 &+ p_n\nabla A_1 + p_{n-2}A_1\\ &+ p_n^2\nabla A_2 + 2p_{n-2}p_nA_2\\ &+ \&c.\end{aligned}$$

Equating the several powers of p_n to zero, we find

$$A_1 = -\frac{1}{p_{n-2}}\nabla A_0,$$

$$2A_2 = -\frac{1}{p_{n-2}}\nabla A_1,$$

$$3A_3 = -\frac{1}{p_{n-2}}\nabla A_2,$$

$$\&c. = \&c.$$

Thus by one regular and easy process each term may be derived from the other.

In performing this process we may omit every term in the subject of operation which does not contain p_{n-2}. For p_{n-2} can be introduced only by performing $\frac{d}{dp_n}$, and since p_n is absent from the coefficients, this operation yields nothing.

In this way we find

$$p_0 Q_2 = p_1,$$
$$p_0^2 Q_3 = p_1 p_2 - p_0 p_3,$$
$$p_0^3 Q_4 = p_1 p_2 p_3 - p_0 p_3^2 - p_1^2 p_4,$$
$$p_0^4 Q_5 = p_1 p_2 p_3 p_4 - p_0 p_3^2 p_4 - p_1^2 p_4^2$$
$$- p_5 (-p_0 p_2 p_3 + p_1 p_2^2 - 2p_0 p_1 p_4)$$
$$+ \frac{p_5^2}{1\,.\,2}(-2p_0^3).$$

To illustrate this process, consider how Q_5 is obtained from Q_4. The first line is formed by multiplying the line above by p_4, this is A_0. To find the coefficient of $-p_5$ we operate with

$$\left(p_3 \frac{d}{dp_5} + p_2 \frac{d}{dp_4}\right) + 2\left(p_1 \frac{d}{dp_3} + p_0 \frac{d}{dp_2}\right)$$

on such of the terms in the line above as contain p_3 and then divide by p_3. Performing the same operation on the coefficient of $(-p_5)$ we obviously obtain the coefficient of $\frac{p_5^2}{1\,.\,2}$.

In M. Serret's *Cours d'Algèbre Supérieure,* Note III., there will be found a method of forming the last term of the equation to the squares of the differences, which suggested the method used in Art. 13, of substituting in a differential equation, if only a differential equation could be found. [See also Dr Salmon's *Higher Algebra,* Arts. 60, 64, and 72.]

CHAPTER III.

Statement of the theory by which the conditions of stability of a dynamical system with n *co-ordinates are made to depend on* 2n *conditions. Arts.* 1—4.

A rule by which these conditions may be derived one from another, together with certain other true but not independent conditions. Arts. 5—8.

A rule by which, when the coefficients of the dynamical equation are letters, the 2n *conditions of stability may be inferred, one from another, without writing down any other conditions. Arts.* 9, 10.

A method by which certain extraneous factors may be discovered and omitted. Arts. 11, 12.

Consideration of the reserved case in which the dynamical equation has equal and opposite roots. Arts. 13—18.

A few geometrical illustrations not necessary to the argument. Arts. 19—25.

Application to a Dynamical Problem. Art. 26.

1. It has been shown in the first Chapter that the stability of a dynamical system with n co-ordinates oscillating about a state of steady motion depends on the nature of the roots of a certain equation of the $2n^{\text{th}}$ degree which we may call

$$f(z) = 0.$$

The system is stable if the real roots and the real parts of the imaginary roots are all negative. Now Cauchy has given the following theorem of which we shall make some use.

Let $z = x + y\sqrt{-1}$ be any root, and let us regard x and y as co-ordinates of a point referred to rectangular axes. Substitute for z and let

$$f(z) = P + Q\sqrt{-1}.$$

Let any point whose co-ordinates are such that P and Q both vanish be called a radical point. Describe any contour, and let a point move round this contour in the positive direction and notice how often $\frac{P}{Q}$ passes through the value zero and changes its sign. Suppose it changes α times from + to − and β times from − to +. Then Cauchy asserts that the number of radical points within the contour is $\frac{1}{2}(\alpha - \beta)$. It is however necessary that no radical point should lie *on* the contour.

2. Let us choose as our contour the infinite semicircle which bounds space on the positive side of the axis of y. Let us first travel from $y = -\infty$ to $y = +\infty$ along the circumference.

If
$$f(z) = p_0 z^n + p_1 z^{n-1} + \dots + p_n,$$
we have changing to polar co-ordinates
$$f(z) = p_0 r^n (\cos n\theta + \sin n\theta \sqrt{-1}) + \dots$$

Hence
$$\left.\begin{aligned} P &= p_0 r^n \cos n\theta + p_1 r^{n-1} \cos (n-1)\theta + \dots \\ Q &= p_0 r^n \sin n\theta + p_1 r^{n-1} \sin (n-1)\theta + \dots \end{aligned}\right\}.$$

In the limit, since r is infinite,
$$\frac{P}{Q} = \cot n\theta;$$

$\frac{P}{Q}$ vanishes when $n\theta = (2\kappa + 1)\frac{\pi}{2}$, *i.e.*
$$\theta = \pm \frac{1}{n}\frac{\pi}{2}, \quad \pm \frac{3}{n}\frac{\pi}{2}, \quad \pm \frac{5}{n}\frac{\pi}{2} \dots\dots\dots\dots\dots\dots (A);$$

$\frac{P}{Q}$ is infinite when $n\theta = 2\kappa\frac{\pi}{2}$, *i.e.*
$$\theta = 0, \quad \pm \frac{2}{n}\frac{\pi}{2}, \quad \pm \frac{4}{n}\frac{\pi}{2}, \quad \pm \frac{6}{n}\frac{\pi}{2} \dots\dots\dots\dots\dots (B).$$

The values of θ in series (B) it will be noticed *separate* those in series (A).

When θ is small and very little greater than zero, $\frac{P}{Q}$ is positive, and therefore changes sign from + to − at every one of the values of θ in series (A). If n be even there will be n changes of sign. If n be odd there will be $n - 1$ changes excluding $\theta = \pm\frac{\pi}{2}$, in this case $\frac{P}{Q}$ is positive when θ is a little less than $\frac{\pi}{2}$, and negative when θ is a little greater than $\frac{\pi}{2}$.

Let us now travel along the axis of y still in the positive direction, viz. from $y=+\infty$ to $y=-\infty$. Since $x=0$ it will be more convenient to use Cartesian co-ordinates, we have, since

$$f(z)=p_0z^n+p_1z^{n-1}+\ldots+p_{n-1}z+p_n,$$

and

$$z=y\sqrt{-1},$$

$$P=p_n-p_{n-2}y^2+p_{n-4}y^4-\ldots$$

$$Q=y\,(p_{n-1}-p_{n-3}y^2+\ldots),$$

and

$$\therefore \frac{P}{Q}=\frac{p_n-p_{n-2}y^2+p_{n-4}y^4-\ldots}{y\,(p_{n-1}-p_{n-3}y^2+\ldots)}.$$

The condition that there should be no radical point within the contour is that this expression should change sign through zero from $-$ to $+$ as often as it before changed sign from $+$ to $-$ on travelling round the semicircle. If n be even the numerator has one more term than the denominator, and when p_0 and p_1 have the same sign, $\frac{P}{Q}$ begins when y is very great by being negative. In order that it should change sign through zero n times, it is necessary and sufficient that both the equations

$$p_n-p_{n-2}y^2+p_{n-4}y^4-\ldots=0,$$

$$p_{n-1}y-p_{n-3}y^3+p_{n-5}y^5-\ldots=0,$$

should have their roots real, and that the roots of the latter should separate the roots of the former.

If n be odd, the numerator and denominator have the same number of terms, and when p_0 and p_1 have the same sign, $\frac{P}{Q}$ begins when y is very great by being positive. In order that it should change sign through zero from $-$ to $+$ $n-1$ times, it is necessary and sufficient that the same two equations as before should have their roots real, and that the roots of the former should separate the roots of the latter.

In order then to express the necessary and sufficient conditions, that $f(z)=0$ *may have no radical point on the positive side of the axis of* y, *put* $z=y\sqrt{-1}$ *and equate to zero separately the real and imaginary parts. Of the two equations thus formed, the roots of the one of lower dimensions must separate the roots of the other. It is also necessary that the coefficients of the two highest powers of* z *in* f (z) *should have the same sign.*

3. It has been stated that p_0 and p_1 the coefficients of the two highest powers in $f(z)$ must have the same sign. It is easy to see

that, if they had opposite signs, $\frac{P}{Q}$ would change sign through zero $2n$ times as we travel round the contour. All the radical points of the equation would then lie on the positive instead of the negative side of the axis of y.

It has also been assumed that no radical point lies on the contour. It has therefore been assumed that $f(z) = 0$ has no root of the form $z = y\sqrt{-1}$. It will be more convenient to consider this exception a little further on.

4. *It is required to express in an analytical form the conditions that the roots of an equation* $f_2(x) = 0$ *may be all real, and may separate the roots of another equation* $f_1(x) = 0$ *of one degree higher dimensions.*

To effect this, let us use Sturm's theorem *reversed.* Perform the process of finding the greatest common measure of $f_1(x)$ and $f_2(x)$, changing the sign of each remainder as it is obtained. Let the series of modified remainders thus obtained be $f_3(x)$, $f_4(x)$, &c. Then it may be shown that when any one of these functions vanishes, the two on each side have opposite signs. It is also clear that no two successive functions can vanish unless $f_1(x)$ and $f_2(x)$ have a common factor. This exception will be considered presently.

Hence in passing from $x = -\infty$ to $+\infty$ no variation of sign can be lost except when $f_1(x)$ vanishes. If a variation is lost it is regained when x has the next greatest value which makes $f_1(x)$ vanish *unless* $f_2(x) = 0$ has a root between these two successive roots of $f_1(x) = 0$. Hence this rule:—

The roots of the equations $f_1(x) = 0$, $f_2(x) = 0$, *will be all real and the roots of the latter will separate those of the former, if in the series*

$$f_1(x),\ f_2(x),\ f_3(x)\ldots$$

as many variations of sign are lost in passing from $x = -\infty$ *to* $x = +\infty$ *as there are units in the degree of the equation* $f_1(x) = 0$.

We have supposed the variations of sign to be *lost* instead of *gained* in passing from $x = -\infty$ to $+\infty$. That this may be the case the *signs* of the highest powers of $f_1(x)$ and $f_2(x)$ must be the same.

These functions are alternately of an even and odd degree, the condition that the whole number of variations of sign may be lost in passing from $x = -\infty$ to $+\infty$ may be more conveniently expressed thus:—*The coefficients of the highest powers of* x *in the series*

$$f_1(x),\ f_2(x),\ f_3(x)\ldots$$

must all have the same sign.

5. The process of finding the greatest common measure of two algebraic expressions is usually rather long. We may in our case shorten it materially by omitting the quotients and performing the division in the following manner. Let

$$f_1(x) = p_0x^n - p_2x^{n-2} + p_4x^{n-4} - \ldots$$
$$f_2(x) = p_1x^{n-1} - p_3x^{n-3} + p_5x^{n-5} - \ldots$$

then, since p_1 is positive, it easily follows by division that

$$f_3(x) = Ax^{n-2} - A'x^{n-4} + A''x^{n-6} - \ldots$$

where

$$A = p_1p_2 - p_0p_3,$$
$$A' = p_1p_4 - p_0p_5,$$
$$\&c. = \&c.,$$

so that by remembering this simple cross-multiplication *we may write down the value of* $f_3(x)$ *without any other process than what may be performed by simple inspection.* In the same way $f_4(x)$, &c. may all be written down.

6. Ex. 1. Express the conditions that the real roots and real parts of the imaginary roots of the cubic

$$x^3 + px^2 + qx + r = 0$$

may be all negative.

$$f_1(x) = x^3 - qx,$$
$$f_2(x) = px^2 - r,$$
$$f_3(x) = (pq - r)\,x,$$
$$f_4(x) = (pq - r)\,r.$$

The necessary conditions are that

$$p,\ pq - r \text{ and } r$$

must all be positive.

Ex. 2. Express the corresponding conditions for the biquadratic

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

$$f_1(x) = x^4 \qquad - qx^2 + s,$$
$$f_2(x) = px^3 \qquad - rx,$$
$$f_3(x) = (pq - r)\,x^2 \qquad - ps,$$
$$f_4(x) = \{(pq - r)\,r - p^2s\}\,x,$$
$$f_5(x) = \{(pq - r)\,r - p^2s\}\,ps.$$

The conditions are that

$$p, \ pq - r, \ (pq - r)\, r - p^2 s \text{ and } s$$

must be all positive.

These are evidently equivalent to the five conditions that

$$p, \ q, \ r, \ s, \ (pq - r)\, r - p^2 s,$$

should be all positive.

In both these examples all the numerical work has been exhibited.

7. Since the coefficients of the highest powers of x in $f_1(x)$ and $f_2(x)$ are p_0 and p_1 we see that the condition that p_0 and p_1 should have the same sign is included in the general statement that all the coefficients of the highest powers should have the same sign. If the function $f(x)$ be of n dimensions we thus obtain n necessary and sufficient conditions.

On examining these conditions in the cases of the cubic and biquadratic it will be seen that they cannot be satisfied if any one of the coefficients of the given equation should be negative.

8. Although the theorem in its present form gives n conditions as the proper number for an equation of the n^{th} degree, yet it is important to notice that it gives other conditions also which are true and may be useful. It has been shown in the second Chapter that all the coefficients of the equation $f(x) = 0$ must be positive, hence the roots of $f_1(x) = 0$ must all be positive. It may be shown also, that the roots of each of the functions $f_1(x)$, $f_2(x)$, &c. are separated by the roots of the function next below it in order. Hence the roots of all these functions must be positive, and therefore in every one of the functions the coefficients of all the powers must be alternately positive and negative and not one can vanish. If however $f_1(x)$ and $f_2(x)$ have one or more common factors some of the functions $f_3(x)$, $f_4(x)$, &c. will wholly vanish.

9. When the degree of the equation is very considerable there is some labour in the application of the rule given in Art. 5. The objection is that we only want the terms in the first column, and to obtain these we have to write down all the other columns. *We shall now investigate a method of obtaining each term in the first column from the one above it without the necessity of writing down any expression except the one required.*

We notice that each function is obtained from the one above it by the same process. Now

$$f_1(x) = p_0 x^n - p_2 x^{n-2} + p_4 x^{n-4} - \dots$$
$$f_2(x) = p_1 x^{n-1} - p_3 x^{n-3} + p_5 x^{n-5} - \dots$$
$$f_3(x) = (p_1 p_2 - p_0 p_3)\, x^{n-2} - (p_1 p_4 - p_0 p_5)\, x^{n-4} + \dots$$

The first and second lines will be changed into the second and third lines by writing for

$$p_0, \qquad p_1, \qquad p_2, \qquad p_3, \quad \&c.$$

the values

$$p_1, \qquad p_1p_2 - p_0p_3, \qquad p_3, \qquad p_1p_4 - p_0p_5, \ \&c.$$

If then in any term of any function we make these changes, we obtain the corresponding term of the function next in order.

10. Example. Express the conditions of stability for the quintic

$$f(x) = p_0x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5.$$

We have

$$f_1(x) = p_0x^5 + \ldots$$

$$f_2(x) = p_1x^4 + \ldots$$

$$f_3(x) = (p_1p_2 - p_0p_3)\,x^3,$$

$$f_4(x) = \{(p_1p_2 - p_0p_3)p_3 - p_1(p_1p_4 - p_0p_5)\}\,x^2,$$

$$f_5(x) = [\{(p_1p_2 - p_0p_3)p_3 - p_1(p_1p_4 - p_0p_5)\}(p_1p_4 - p_0p_5) - (p_1p_2 - p_0p_3)^2 \cdot p_5]\,x,$$

$$f_6(x) = [\{(p_1p_2 - p_0p_3)p_3 - p_1(p_1p_4 - p_0p_5)\}(p_1p_4 - p_0p_5) - (p_1p_2 - p_0p_3)^2 \cdot p_5](p_1p_2 - p_0p_3)p_5.$$

11. On examining the conditions as given in the cases of a biquadratic and quintic it will be apparent that several contain the previous conditions as factors. Thus the analytical expressions are rendered much longer than is necessary. *It is now proposed to investigate a method of discovering and omitting these extraneous factors as they occur, and thus obtaining the required conditions in their simplest forms.*

Let the coefficients of the several powers of x in the functions be when taken positively

$$\begin{array}{llll} f_1(x) = p_0, & p_2, & p_4, & p_6 \cdots \\ f_2(x) = p_1, & p_3, & p_5, & p_7 \cdots \\ f_3(x) = A, & A', & A'', & A''' \ldots \\ f_4(x) = B, & B', & B'', & B''' \ldots \\ & \&c. = \&c. & & \end{array}$$

Let us first find which of these terms contain p_1 as a factor. Putting $p_1 = 0$ and using the rule in Art. 5, the series become

$$\begin{array}{cccccc}
p_0, & p_2, & p_4, & p_6, & \dots & \dots \\
0, & p_3, & p_5, & p_7, & \dots & \dots \\
-p_0p_3, & -p_0p_5, & -p_0p_7, & -p_0p_9, & \dots & \dots \\
-p_0p_3^2, & -p_0p_3p_5, & -p_0p_3p_7, & -p_0p_3p_9, & \dots & \dots \\
0, & 0, & 0, & 0, & \dots & \dots \\
0, & 0, & 0, & 0, & \dots & \dots \\
 & & \&c. & & &
\end{array}$$

Hence the C's and D's all vanish and therefore contain p_1 as a factor. By the rule in Art. 5, the E's contain p_1^2, the F's contain p_1^3, the G's p_1^5, the H's p_1^8, and so on.

But since each line is formed from the preceding by a uniform rule, it follows that the D's and E's contain A as a factor, the F's contain A^2, the G's contain A^3, the H's contain A^5, and so on.

The factor A in the D's and E's has its origin in the factor p_1 which occurs in the C's and D's and would not appear if that factor had been omitted when the C's and D's were formed. The factor A in the D's and E's in the same way gives rise to the factor B in the E's and F's. So that if we take care each time we perform the process described in Art. 9 to omit the common factor p_1 whenever it occurs, all these subsequent factors will never make their appearance.

We shall now show that if these factors are omitted, the dimensions of the n^{th} function $f_n(x)$ will be $n\dfrac{n-1}{2}$. First consider the actual dimensions of each function before the factors are omitted. If we examine the rule by which each function is derived from the preceding, it will become evident that, the dimensions of each letter being indicated by its suffix, the dimensions of any function are equal to the sum of the two preceding $+2$.

In the following table the first column indicates the function. In the second column will be found the dimensions of the leading coefficient of that function when calculated by the rule in Art. 5. In the third column will be found the dimensions as given by the formula $n\dfrac{n-1}{2}$. In the remaining columns are the dimensions of the extraneous factors p_1, A, B, &c. introduced into each term.

$f_1(x)$	0	0						
$f_2(x)$	1	1						
$f_3(x)$	3	3						
$f_4(x)$	6	6						
$f_5(x)$	11	10	1					
$f_6(x)$	19	15	1	3				
$f_7(x)$	32	21	2	3	6			
$f_8(x)$	53	28	3	6	6	10		
$f_9(x)$	87	36	5	9	12	10	15	
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

Each term in the second column is the sum of the two terms just above it $+2$. The n^{th} term in the third column is equal to the term just above it $+(n-1)$. In all the other columns each term is the sum of the two terms just above it. The last term in the n^{th} row is equal to the $(n-3)^{\text{th}}$ term in the third column. We wish to show that any term in the second column is equal to the sum of all the terms in the same row to the right of that term. It is not difficult to show from the data just given that if this be true for any two adjacent rows, it is true for all the others, and hence we may assume it to be always true.

It is clear that these extraneous factors may be omitted since by the conditions already expressed they are all positive. When omitted as they occur, the dimensions of the n^{th} function has just been shown to be $n\frac{n-1}{2}$. It is easy to see that the conditions thus reduced must contain the terms

$$p_1, \quad p_1p_2, \quad p_1p_2p_3, \quad p_1p_2p_3p_4, \text{ \&c.}$$

Now if we take any one of these as

$$p_1p_2p_3p_4p_5p_6 \dots\dots (1),$$

and operate by the rule in Art. 9, we have

$$(p_1p_2 - p_0p_3)p_3(p_1p_4 - p_0p_5)p_5(p_1p_6 - p_0p_7)p_7,$$

which contains the term

$$\overline{p_1p_2}p_3\overline{p_1p_4}p_5\overline{p_1p_6}p_7 \dots\dots (2).$$

Thus we have p_1 introduced as often as there is a factor in (1) with an odd suffix. But it should be introduced only once. These extra p_1's are the extraneous factors to be omitted. Each of these, if left, would appear as the factor $p_1p_2 - p_0p_3$ in the next condition, and be still more complicated in the next after that.

In order then to obtain the several conditions in their simplest form it is only necessary after performing the operation described in Art. 5 or Art. 9 to divide by p_1^κ, where κ is one less than the number of factors with odd suffixes in the condition operated on.

12. Example. Express the conditions of stability for the sextic

$$f(x) = p_0x^6 + p_1x^5 + p_2x^4 + p_3x^3 + p_4x^2 + p_5x + p_6 = 0.$$

We have

$$f_1(x) = p_0x^6 + \ldots$$
$$f_2(x) = p_1x^5 + \ldots$$
$$f_3(x) = (p_1p_2 - p_0p_3)x^4 + \ldots$$
$$f_4(x) = (p_1p_2p_3 - p_0p_3^2 - p_1^2p_4 + p_0p_1p_5)x^2 + \ldots$$
$$f_5(x) = \{p_1p_2p_3p_4 - p_0p_3^2p_4 - p_1^2p_4^2 + 2p_0p_1p_4p_5 - p_1p_2^2p_5 + p_0p_2p_3p_5 - p_0^2p_5^2 + p_1^2p_2p_6 - p_0p_1p_3p_6\}x^2 + \ldots$$
$$f_6(x) = [p_1p_2p_3p_4p_5 - p_0p_3^2p_4p_5 - p_1^2p_4^2p_5 + 2p_0p_1p_4p_5^2 - p_1p_2^2p_5^2 + p_0p_2p_3p_5^2 - p_0^2p_5^3 + 2p_1^2p_2p_5p_6 - 3p_0p_1p_3p_5p_6 - p_1p_2p_3^2p_6 + p_0p_3^3p_6 + p_1^2p_3p_4p_6 - p_1^3p_6^2]x + \&c.$$
$$f_7(x) = \text{coefficient of } x \text{ in } f_6\, x \times \text{by } p_6.$$

13. In the preceding theory two reservations have been made.

1. In applying Cauchy's theorem it has been assumed that there were no radical points on the axis of y.

2. It has been assumed that P and Q have no common factor, so that none of the functions f_1, f_2, &c. vanish absolutely.

If any radical point lie on the axis of y, it is clear that $f(z) = 0$ must have a factor of the form $(z^2 + a^2)^r$. Let $f(z) = (z^2 + a^2)^r \phi(z)$. In this case when we put $z = y\sqrt{-1}$, we have

$$f(z) = (a^2 - y^2)^r (P' + Q'\sqrt{-1});$$
$$\therefore \left.\begin{aligned} P &= (a^2 - y^2)^r P' \\ Q &= (a^2 - y^2)^r Q' \end{aligned}\right\}.$$

Thus P and Q have a common factor, and we are warned of the possible existence of radical points on the contour by the total vanishing of some one of the functions f_1, f_2, f_3, &c.

The two reserved cases may therefore be included in one. If $f(z) = 0$ be the equation furnished by dynamical considerations, we form the functions f_1, f_2, f_3, &c. If all these be finite, the question of the stability of the system has been answered. If any one vanish absolutely, $f_1(x)$ and $f_2(x)$ have a common measure, and we must add some further considerations. It will be convenient to examine separately the dynamical effect of the roots which do not and which do enter through the greatest common measure. Let us begin with the former.

14. Following the same notation as before, we have

$$f(z) = p_0 z^n + p_1 z^{n-1} + \ldots + p_{n-1} z + p_n,$$

$$\left. \begin{aligned} P &= \pm f_1(y) = p_n - p_{n-2} y^2 + \ldots\ldots \\ Q &= \pm f_2(y) = p_{n-1} y - p_{n-3} y^3 + \ldots \end{aligned} \right\}.$$

If then $f(z)$ have two roots, viz. $\pm(h + k\sqrt{-1})$, which are equal and opposite, then $f_1(y)$ and $f_2(y)$ must have two common roots, viz. $\pm \dfrac{h + k\sqrt{-1}}{\sqrt{-1}}$. The common measure therefore of $f_1(y)$ and $f_2(y)$ contains all the roots of $f(y\sqrt{-1})$ which are equal and opposite. Conversely the greatest common measure of P and Q is necessarily an even function of y, and if it be equated to zero, its roots are necessarily equal and opposite. These roots must also satisfy $f(y\sqrt{-1}) = 0$.

Let this greatest common measure be $\psi(y^2) = 0$, and let y^{2r} be the highest power which enters into it. Also let

$$f(z) = \psi(-z^2)\,\phi(z),$$

then $\phi(z)$ is a function which, as has just been shown, has not got two roots equal and opposite, and to this function we may apply Cauchy's theorem without fear of failure. Putting $z = y\sqrt{-1}$, let

$$\phi(z) = P' + Q'\sqrt{-1}.$$

Then we wish to express the condition that $\dfrac{P'}{Q'}$ should change sign from $-$ to $+$ through zero $n - 2r$ times if n be even and $n - 2r - 1$ times if n be odd. But

$$f(y\sqrt{-1}) = P + Q\sqrt{-1},$$

and $$f(y\sqrt{-1}) = \psi(y^2)\,(P' + Q'\sqrt{-1});$$

$$\therefore \quad \left. \begin{aligned} P &= \psi(y^2)\,P' \\ Q &= \psi(y^2)\,Q' \end{aligned} \right\}.$$

Thus the number of changes of sign in $\dfrac{P'}{Q'}$ is exactly the same as that of $\dfrac{P}{Q}$. The factor $\psi(y^2)$ will run through all the functions $f_2(y)$, $f_3(y)$, &c. obtained from $f_1(y)$ by a process which is equivalent to that of finding the greatest common measure of $f_1(y)$ and

$f_2(y)$. The changes of sign of this factor will therefore not affect the number of variations of sign in the series f_1, f_2, f_3, &c.

The last factor which is not zero is $f_{n+1-2r}(y)$ if n be the dimensions of $f_1(y)$.

Hence *if we omit the considerations of the vanishing factors and apply the same rule as before to the* n + 1 − 2r *remaining factors, we can express the condition that the proper number of changes of sign from* − *to* + *have been lost through zero in the function* ϕ (z), *i.e. that the roots not given by the vanishing of* f_{n+1-2r} *are all of the character to ensure stability.*

15. Let us next consider the effect on stability of the roots indicated by the absolute vanishing of one of the subsidiary functions. This function must be of the form

$$\psi(y^2) = q_0 y^n - q_2 y^{n-2} + \dots,$$

where n is even. The corresponding factor of $f(z)$ is

$$F(z) = q_n + q_{n-2}z^2 + q_{n-4}z^4 + \dots.$$

The roots of this equation are two and two equal with opposite signs, it is therefore necessary for stability that no root should have any real part. To express this condition, draw a straight line parallel to the axis of y at an indefinitely short distance from it, viz. $x = h$. Let us apply, in the same manner as before, Cauchy's theorem to the contour formed by this straight line and the infinite semicircle on its positive side. Putting $z = h + z'$, we have

$$F(z) = q_n + 2q_{n-2}hz' + q_{n-2}z'^2 + 4q_{n-4}hz'^3 + \dots$$

The two functions are therefore, omitting the positive factor h,

$$P = q_n - q_{n-2}y^2 + q_{n-4}y^4 - \&c.,$$

$$Q = 2q_{n-2}y - 4q_{n-4}y^3 + \&c.$$

Now $f_1(y)$ and $f_2(y)$ are what P and Q become when arranged in descending powers of y and the coefficients of their highest powers *made to have the same sign*. Hence

$$f_2(y) = \frac{df_1(y)}{dy}.$$

The rule described in Art. 4 will now become the same as that usually called *Sturm's theorem*. We are to seek the greatest common measure of $f_1(y)$ and its differential coefficient, and make the coefficients of the highest powers of y in these two and in the series of modified remainders all positive.

That we should have been led to Sturm's theorem in this case is just what we might have expected. For to express the conditions that the roots of

$$F(z) = q_n + q_{n-2}z^2 + q_{n-4}z^4 + \dots$$

are all of the form $\pm r\sqrt{-1}$ is the same thing as to express the conditions that the roots of

$$q_n - q_{n-2}z^2 + q_{n-4}z^4 - \dots = 0$$

are all real.

16. There is however another mode of proceeding. Suppose we have calculated the functions f_1, f_2, &c. for the general equation

$$f(z) = p_0z^n + p_1z^{n-1} + \dots$$

and find when the values of p_0, p_1, &c. are substituted that some one function say f_r of the series absolutely vanishes, and therefore also all the functions which follow it. Then operate on each of these vanishing functions with

$$p_{n-1}\frac{d}{dp_n} + 2p_{n-2}\frac{d}{dp_{n-1}} + 3p_{n-3}\frac{d}{dp_{n-2}} + \dots$$

repeating the operation until we obtain a result which is not zero. If we now replace these vanishing functions by these results we may apply the rule of Art. 4, just as if these were the functions supplied by the process of the greatest common measure. As this process is not so convenient as that already given it is unnecessary to consider it in detail*.

17. As a numerical example, let us examine whether the roots of

$$f(x) = x^8 + 2x^7 + 4x^6 + 4x^5 + 6x^4 + 6x^3 + 7x^2 + 4x + 2 = 0$$

satisfy the conditions of stability. In order to show the working of the method it will be necessary to exhibit all the numerical calculations. We have by Art. 5,

* The function $f_r(x)$ vanishes because the equation $f(z) = 0$ has two roots equal and opposite. If we put $z = z' + h$, where h is as small as we please, this peculiarity will disappear. Thus if the values of z are of the form $\pm(\alpha \pm \beta\sqrt{-1})$ the corresponding values of z' are $-h \pm \alpha \pm \beta\sqrt{-1}$. These values of z will indicate stability if α be zero and instability if α have any value positive or negative. If h be as small as we please and positive, the values of z' will indicate stability or instability under the same circumstances. We may therefore apply the rule of Art. 4 to the function $f(z' + h)$ instead of $f(z)$, provided we retain only the lowest powers of h which occur. Hence all the functions $f_1(x)$, $f_2(x) \dots f_{r-1}(x)$ which do not vanish are unaltered. To find what function will replace $f_r(x)$ we must increase by h all the roots of $f(z) = 0$ when their signs have been changed. This may be effected by performing on $f(z)$ the operation represented by Δ in Art. 7. The rule in the text therefore follows from the one given in that article.

$$f_1(x) = x^8 - 4x^6 + 6x^4 - 7x^2 + 2,$$
$$f_2(x) = 2x^7 - 4x^5 + 6x^3 - 4x,$$
$$f_3(x) = 4x^6 - 6x^4 + 10x^2 - 4,$$
$$f_4(x) = 4x^5 - 4x^3 + 8x,$$
$$f_5(x) = 8x^4 - 8x^2 + 16.$$

Here we find $f_6(x)$ to be absolutely zero, accordingly by Art. 15 we replace it by the differential coefficient of $f_5(x)$, this being Sturm's rule. We have therefore

$$f_6(x) = 8\,(4x^3 - 2x),$$
$$f_7(x) = 8^2\,(2x^2 - 6),$$
$$f_8(x) = -8^3 . 20 . x,$$
$$f_9(x) = -8^5 . 120.$$

We see that the two last of the coefficients of the highest powers are negative. The roots therefore do *not* satisfy the conditions of stability.

As another example, take the equation

$$f(x) = x^6 + x^5 + 6x^4 + 5x^3 + 11x^2 + 6x + 6.$$

Here

$$f_1(x) = x^6 - 6x^4 + 11x^2 - 6,$$
$$f_2(x) = x^5 - 5x^3 + 6x,$$
$$f_3(x) = x^4 - 5x^2 + 6,$$
$$f_4(x) = 0.$$

Replacing $f_4(x)$ by the differential coefficient of $f_3(x)$, we have

$$f_4(x) = 4x^3 - 10x,$$
$$f_5(x) = 10x^2 - 24x,$$
$$f_6(x) = 4x,$$
$$f_7(x) = 4 . 24.$$

Here all the coefficients of the highest powers are positive, hence the roots satisfy the conditions of stability.

It is clear that when the coefficients are *numerical* the rule given in Art. 5 is the most convenient, but when the coefficients are letters, the rule in Art. 9 will be found preferable.

The process would be simplified by omitting the alternate positive and negative signs of the terms in each line.

18. It may be interesting *to express the two subsidiary functions* f_1 (x) *and* f_2 (x) *in terms of the roots of the given equation.*

Let $a_1, a_2, a_3 \ldots a_n$ be the roots of the given equation $f(x) = 0$, so that

$$f(x) = (x - a_1)(x - a_2)(x - a_3)\ldots$$
$$= x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots$$

Then it is evident that

$$\pm f_1 (x\sqrt{-1}) = x^n + p_2 x^{n-2} + p_4 x^{n-4} + \ldots$$
$$= \tfrac{1}{2}(x + a_1)(x + a_2)\ldots + \tfrac{1}{2}(x - a_1)(x - a_2)\ldots$$

It may be shown* that

$$\pm f_2 (x\sqrt{-1}) = p_1 x^{n-1} + p_3 x^{n-3} + \ldots$$
$$= -\Sigma \frac{(a_1 + a_2)(a_1 + a_3)\ldots(a_1 + a_n)}{(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n)} a_1 (x - a_2)\ldots(x - a_n).$$

19. The following propositions are not necessary to the main argument, but as they illustrate geometrically the propositions in this chapter it has been considered proper to state them very briefly. The demonstrations will therefore be much curtailed.

The equation being

$$f(z) = p_0 z^n + p_1 z^{n-1} + \ldots + p_{n-1} z + p_n = 0,$$

we put as in Art. 1, $z = x + y\sqrt{-1}$, and thus obtain two curves, whose equations expressed in polar co-ordinates are

$$\left.\begin{aligned} P &= p_0 r^n \cos n\theta + p_1 r^{n-1} \cos (n-1)\theta + \ldots = 0 \\ Q &= p_0 r^n \sin n\theta + p_1 r^{n-1} \sin (n-1)\theta + \ldots = 0 \end{aligned}\right\}.$$

These intersect in the radical points of the equation $f(z) = 0$.

20. If we trace these curves we find that the curve $P = 0$ has n asymptotes whose *directions* are given by $\cos n\theta = 0$, *i.e.*

$$\theta = \frac{1}{n}\frac{\pi}{2}, \quad \frac{3}{n}\frac{\pi}{2}, \quad \frac{5}{n}\frac{\pi}{2}, \ldots\ldots$$

* Let us assume

$$p_1 x^{n-1} + p_3 x^{n-3} + \&c. = A_1 (x - a_2)\ldots(x - a_n) + A_2 (x - a_1)\ldots(x - a_n) + \&c.,$$

where A_1, A_2, &c. are constants whose values have to be found. Putting $x = a_1$, we have $p_1 a_1^{n-1} + p_3 a_1^{n-3} + \&c. = A_1 (a_1 - a_2)\ldots(a_1 - a_n)$.

But since $x^n + p_1 x^{n-1} + \ldots = (x - a_1)(x - a_2)\ldots(x - a_n)$,

we have by putting $x = a_1$ and $x = -a_1$

$$a_1^n + p_1 a_1^{n-1} + \ldots = 0,$$
$$a_1^n - p_1 a_1^{n-1} + \ldots = 2a_1 (a_1 + a_2)\ldots(a_1 + a_n).$$

Subtracting the second of these results from the first we find A_1 to have the value given in the text.

These asymptotes all pass through the same point on the axis of x, viz. $x = -\frac{p_1}{np_0}$. It is also clear that only one branch of the curve can go to each end of an asymptote. Similar remarks apply to the curve $Q = 0$, the directions of its asymptotes being given by $\sin n\theta = 0$, *i.e.*

$$\theta = 0,\quad \frac{2}{n}\frac{\pi}{2},\quad \frac{4}{n}\frac{\pi}{2},\quad \frac{6}{n}\frac{\pi}{2},\ldots\ldots$$

21. From these simple propositions we might, if it were worth while, deduce that every equation must have a root. The asymptotes of the two curves $P = 0$, $Q = 0$ are *alternate*, and no two branches of the same curve can approach the same end of an asymptote. By sketching a figure, it may be easily shown that some branch of the P curve must cut some branch of the Q curve.

22. Let us next consider the intersections of the curves $P = 0$, $Q = 0$.

If we transform the origin to h, k, we put $x = h + \xi$, $y = k + \eta$. This is the same as expanding $f(h + k\sqrt{-1} + \xi + \eta\sqrt{-1})$, and collecting into two parcels the real and imaginary terms. Let the expansion be

$$A_0 + A_1(\xi + \eta\sqrt{-1}) + A_2(\xi + \eta\sqrt{-1})^2 + \ldots$$

where $A_0 A_1 \ldots$ are of the form

$$c(\cos\alpha + \sin\alpha\sqrt{-1}).$$

If we put $\xi + \eta\sqrt{-1} = r(\cos\theta + \sin\theta\sqrt{-1})$, we have

$$\left.\begin{array}{l} P = c_0\cos\alpha_0 + c_1 r\cos(\theta + \alpha_1) + c_2 r^2\cos(2\theta + \alpha_2) + \ldots \\ Q = c_0\sin\alpha_0 + c_1 r\sin(\theta + \alpha_1) + c_2 r^2\sin(2\theta + \alpha_2) + \ldots \end{array}\right\}.$$

If the point (h, k) be a point of intersection $c_0 = 0$. If the intersection be a double point on either curve, the terms of the first degree must be zero, therefore $c_1 = 0$, and the origin is therefore a double point on the other curve also.

It is not difficult to show that if the intersection be a multiple point of any degree of multiplicity on one curve, it is a point of the same degree of multiplicity on the other curve. The tangents to these branches all make equal angles with each other, the tangents to the P and Q curves being *alternate* as we travel round the point of intersection. If the intersection be not a multiple point on either curve, the branches cut at right angles.

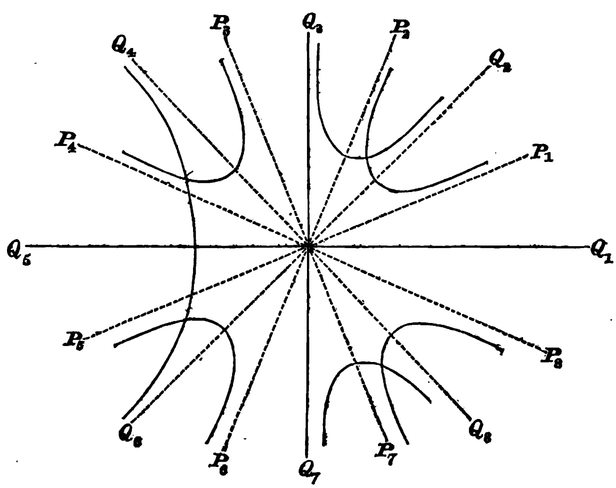
Let us travel round a point of intersection along the circumference of a small circle whose centre is the point of intersection in the direction in which θ is measured. Then it may be shown that as we pass from a P curve to a Q curve, P and Q have opposite signs, and as we pass from a Q curve to a P curve the same sign. This is in fact merely Cauchy's rule for the changes of sign of $\frac{P}{Q}$ when we travel round a radical point.

23. Let us express the condition that there is no radical point on the positive side of the axis of y. This is the geometrical illustration of Art. 2.

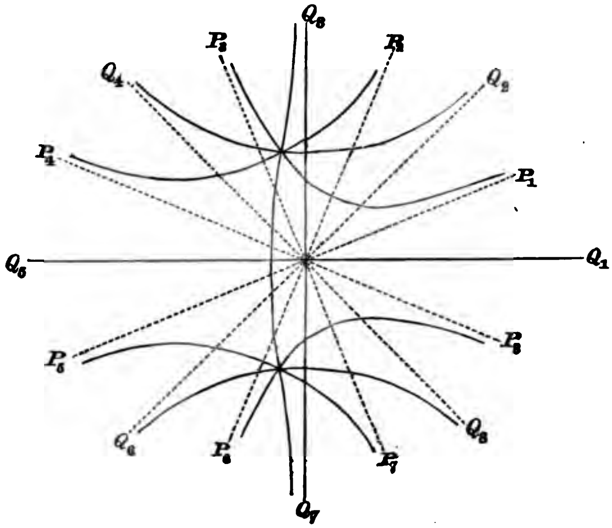
Draw a circle of infinite radius, and let it cut the asymptotes of the P curve in P_1, P_2, P_3,...P_{2n} and the asymptotes of the Q curve in Q_1, Q_2,...Q_{2n}. These points alternate with each other. Taking only those points which lie on the positive side of the axis of y, the P and Q curves may be said to begin at these points and are to intersect each other only on the negative side of the axis of y. The branches of the two curves must therefore remain alternate with each other throughout the space on the positive side of the axis of y. Their points of intersection with the axis of y must be also alternate, and hence if we put $x=0$, in the equations $P=0$, $Q=0$, and regard them as equations to find y the roots of each must separate the roots of the other.

Conversely, we may show that if the intersections of the two branches are alternate on the axis of y, they cannot have intersected on that side of the axis of y on which the common intersection of all the asymptotes is not. This is the result arrived at in Art. 2.

24. The following diagrams exhibit the forms of the curve $P=0$, $Q=0$ for a biquadratic. The dotted lines represent the asymptotes.



A biquadratic with four imaginary roots.



A biquadratic with equal imaginary roots.

25. [If $z = x + y\sqrt{-1}$, we have

$$f(z) = P + Q\sqrt{-1}.$$

Differentiating with respect, firstly to x and secondly to y, we find

$$\left.\begin{aligned} f'(z) &= \frac{dP}{dx} + \frac{dQ}{dx}\sqrt{-1} \\ f'(z) &= \frac{dQ}{dy} - \frac{dP}{dy}\sqrt{-1} \end{aligned}\right\}.$$

It easily follows that $\dfrac{dP}{dx} = \dfrac{dQ}{dy}$ and $\dfrac{dQ}{dx} = -\dfrac{dP}{dy}$; so that both the functions P and Q satisfy the equation

$$\frac{d^2V}{dy^2} + \frac{d^2V}{dx^2} = 0.$$

The equation $f(z) = 0$ gave us two curves which we have called $P = 0$ and $Q = 0$. In the same way the derived equation $f'(z) = 0$ will give us two other curves, which we may represent by $P' = 0$ and $Q' = 0$. These we may call the *derived* P and Q curves.

If as in Art. 22 we transform the origin to the point (h, k) we have

$$P' = c_1 \cos \alpha_1 + 2c_2 r \cos(\theta + \alpha_2) + 3c_3 r^2 \cos(2\theta + \alpha_3) + \dots$$
$$Q' = c_1 \sin \alpha_1 + 2c_2 r \sin(\theta + \alpha_2) + 3c_3 r^2 \sin(2\theta + \alpha_3) + \dots$$

If the origin be at a point of intersection of the curves $P = 0$, $P' = 0$ which is not a double point on the first of these curves, we have $c_0 \cos \alpha_0 = 0$ and $\cos \alpha_1 = 0$. Hence the tangent to the curve $P=0$ at the point of intersection is parallel to the axis of x. Conversely if we move the origin to any point on the curve $P=0$ at which the tangent is parallel to the axis of x, we find that the curve $P'=0$ passes through the origin. Hence the derived P curve passes through all those points on the curve $P=0$ at which the tangent is parallel to the axis of x, and all those points on the curve $Q=0$ at which the tangent is parallel to the axis of y. In the same way the derived Q curve passes through those points on the curve $Q=0$ at which the tangent is parallel to the axis of x, and those points on the curve $P=0$ at which the tangent is parallel to the axis of y.

If $c_0 \cos \alpha_0 = 0$, and $c_1 = 0$, the origin is a double point on the curve $P=0$ and the origin also lies on both the curves $P'=0$, $Q'=0$. So that both the derived curves cut the curves $P=0$ and $Q=0$ in their double points. In other words, these double points are radical points for the derived equation $f'(z)=0$. In the same way, any multiple point on either curve is a multiple point of one degree less multiplicity on both the derived curves.

If a finite straight line AB be drawn parallel to the axis of x joining two points A, B on the same or on different branches of the curve $P=0$, this finite straight line must cut one or more branches of the derived P curve. For it is clear that if P vanishes at A and B, P' which is equal to $\frac{dP}{dx}$ cannot keep one sign between A and B, and must therefore vanish somewhere between A and B. If A and B be adjacent points, *i.e.* if there be no other points between A and B belonging to the curve $P=0$, then the straight line AB must cut an *odd* number of branches of the derived P curve. In the same way if a straight line CD be drawn parallel to the axis of y joining two adjacent points on a derived Q curve, this straight line must also cut an odd number of branches of a derived P curve between C and D.

By considering a tangent as the limit of secant, it again follows that if a tangent be drawn to the curve $P=0$ parallel to the axis of x, the derived P curve must pass through the point of contact.

Let R be a radical point on the derived curve $f'(z)=0$, and let it not be a double point on either of the curves $P=0$, $Q=0$. Let a straight line be drawn from R in any direction cutting the branches of either of the curves $P=0$, $Q=0$ in the points A_1, A_2, &c. Then we may show that

$$\frac{1}{RA_1} + \frac{1}{RA_2} + \frac{1}{RA_3} + \text{\&c.} = 0,$$

so that the polar line of R with regard to either of the curves $P=0$, $Q=0$ is at infinity.

The positions of the radical points of the derived equation $f'(z)=0$ relatively to any branch or branches of the curves $P=0$, $Q=0$ may be found by the use of Cauchy's theorem. If the point spoken of in Art. 1 travel along a branch of the curve $P=0$, it is easy to see that $\frac{P'}{Q'}=\frac{dy}{dx}$. If it travel along a branch of the curve $Q=0$ we have $\frac{P'}{Q'}=-\frac{dx}{dy}$. If then any contour be partly bounded by branches of these curves, the simplest inspection of the points at which the tangents are parallel to the axis will determine the changes of sign of $\frac{P'}{Q'}$ as it passes through zero. If another part of the contour be an arc of a circle of infinite radius whose centre is the origin, the changes of sign through zero will be from + to − and their number will be indicated by the number of asymptotes of the derived P curve which cut the arc.]

26. The use of Watt's Governor in the steam engine is too well known to need description. It has however, as commonly used, a great defect. It is sometimes of importance that the engine should continue to work at the same rate notwithstanding great changes in the resistances. Suppose the load suddenly diminished, the engine works quicker, the balls diverging cut off the steam, and the engine, after a time, again works uniformly, but at *a different rate from before.* The balls as they open out or close in are usually made to describe circles. Let them now be constrained to describe some other curve which we may afterwards choose so as to correct the above defect. If this curve be a parabola and the balls be treated as particles, it is clear from very elementary considerations that these will be in relative equilibrium only when the engine works at a given rate. This principle is due to Huyghens, see *Astronomical Notices*, December, 1875. It is now proposed to determine the condition of stable oscillation about a state of steady motion.

Two equal rods AB, AB' are attached at A by hinges to a small ring which can slide smoothly along a vertical axis. The ring is attached by a rod to the valve and can thus govern the amount of steam admitted. Two equal balls are attached at B and B', and the centre of gravity G of the rod AB and the ball B is constrained to describe some curve. To represent the inertia of the engine we shall suppose a horizontal fly-wheel attached to the vertical axis whose moment of inertia about the axis is I. Let the excess of the action of the steam over the resistance of the load be represented by some couple whose moment about the vertical axis is $f(\theta)$, where θ is the inclination of the rod AG to the vertical, and f is a function which depends on the construction of the engine. Since the steam is cut off when the balls open out, it is clear that $\frac{df(\theta)}{d\theta}$ is negative.

There may be also some resistances which vary with the velocity. Let these be represented by a couple $B\frac{d\phi}{dt}$ tending to retard the motion round the vertical axis and a couple round A in the plane BAB' equal to $mC\frac{d\theta}{dt}$, where ϕ is the angle the vertical plane BAB' makes with a fixed vertical plane.

Let m be the mass of either sphere and rod; k the radius of gyration about an axis through G perpendicular to the rod, and k' that about the rod, let $l = AG$. Then the equation of angular momentum gives

$$\frac{d}{dt}\left\{I\frac{d\phi}{dt} + 2m\left(\overline{k^2+l^2}\sin^2\theta + k'^2\cos^2\theta\right)\frac{d\phi}{dt}\right\} = f(\theta) - B\frac{d\phi}{dt}.$$

Let the steady motion be given by $\theta = \alpha$, $\frac{d\phi}{dt} = n$, and let the oscillations be represented by $\theta = \alpha + x$, $\frac{d\phi}{dt} = n + y$. We have then

$$f(\theta) = f(\alpha) + f'(\alpha)\,x,$$
$$f(\alpha) = Bn.$$

The equation then reduces to

$$\{I + 2m(\overline{k^2+l^2}\sin^2\alpha + k'^2\cos^2\alpha)\}\frac{dy}{dt} + 2m(l^2+k^2-k'^2)\sin 2\alpha n\frac{dx}{dt} = f'(\alpha)x - By.$$

This equation may be briefly written

$$A\frac{dy}{dt} + By + E\frac{dx}{dt} + Fx = 0.$$

Let z be the altitude of G above some fixed horizontal plane. Then if T be the semi vis viva

$$2T = I\phi'^2 + 2m\{(k^2+l^2)\sin^2\theta + k'^2\cos^2\theta\}\phi'^2 + 2mk^2\theta'^2 + 2m\left\{\left(\frac{dz}{d\theta}\right)^2 + l^2\cos^2\theta\right\}\theta'^2.$$

If U be the force function, omitting the couple of resistance, we have $U = -2mgz$. The virtual moment of the couples of resistance being $2mC\theta'\delta\theta$, the Lagrangian equation of motion becomes

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta} - 2mC\theta'.$$

Substituting for T and U we have when the motion is steady

$$\frac{n^2}{g}(l^2+k^2-k'^2)\sin\theta\cos\theta=\frac{dz}{d\theta},$$

$$\therefore \frac{n^2}{2g}(l^2+k^2-k'^2)\sin^2\theta=z.$$

Since $l\sin\theta$ is the distance of G from the vertical axis, we see that the path of G must be a parabola. The semi latus rectum is

$$\frac{gl^2}{n^2(l^2+k^2-k'^2)},$$

which, we notice, is independent of the radius of the balls. The length of this latus rectum must of course be adjusted to suit the particular rate at which the engine is intended to work.

When the system is oscillating about the state of steady motion we have, putting

$$a^2=l^2+\frac{n^4}{g^2}(l^2+k^2-k'^2)^2\sin^2\alpha,$$

and rejecting the squares of x and y,

$$(k^2+a^2\cos^2\alpha)\frac{d^2x}{dt^2}+C\frac{dx}{dt}-(l^2+k^2-k'^2)\sin 2\alpha ny=0.$$

The term x, it will be noticed, has disappeared from the equation. This equation may be briefly written in the form

$$H\frac{d^2x}{dt^2}+C\frac{dx}{dt}-Ly=0.$$

Eliminating y from the equations of motion we have

$$AH\frac{d^3x}{dt^3}+(AC+BH)\frac{d^2x}{dt^2}+(BC+EL)\frac{dx}{dt}+FLx=0.$$

The coefficients are all positive, the necessary and sufficient condition of stability is therefore

$$(AC+BH)(BC+EL)>AFHL.$$

In some clocks to which Watt's Governor is applied, there is a special arrangement which causes C to be much greater than B. See the *Astronomical Notes*, XI., 1851, and the *Memoirs of the Astronomical Society*, Vol. XX. Neglecting therefore B, we have

$$CE>FH.$$

$$\therefore 2Cmn(l^2+k^2-k'^2)\sin 2\alpha>F(k^2+a^2\cos^2\alpha).$$

CHAPTER IV.

Formation of the equations of steady motion and of small oscillation where Lagrange's method may be used. *Arts.* 1—5.

The equations being all linear the conditions of stability are expressed by the character of the roots of a determinantal equation of an even order. *Art.* 6.

Mode of expanding the determinant. *Art.* 7.

A method of finding the proper co-ordinates to make the coefficients of the Lagrangian function constant. *Arts.* 8—10.

How the Harmonic oscillations about steady motion differ from those about a position of equilibrium. The forces which cause the difference are of the nature of centrifugal forces produced by an imaginary rotation about a fixed straight line. *Arts.* 11—19.

Reduction of the fundamental determinant to one of fewer rows by the elimination of all co-ordinates which do not appear except as differential coefficients in the Lagrangian function; with an example. *Arts.* 20—23.

Formation of the equations of Motion and of the determinant when the geometrical equations contain differential coefficients, so that Lagrange's method cannot be used; with an example. *Arts.* 24—27.

1. Let the system be referred to any co-ordinates ξ, η, ζ, &c. The general expression for the kinetic energy is

$$T = \tfrac{1}{2} P\xi'^2 + Q\xi'\eta' + \dots$$

where P, Q, &c. are known functions of ξ, η, ζ, &c. and accents have their usual meaning. Let us suppose the system to have some motion represented by

$$\xi = f(t), \quad \eta = F(t), \quad \text{\&c.}$$

and when disturbed, we wish to find the oscillations about this motion. To effect this, we put

$$\xi = f(t) + \theta, \quad \eta = F(t) + \phi, \quad \text{\&c.}$$

where θ, ϕ, &c. are all small quantities. Substituting and expanding T in powers of θ, ϕ, &c., we find

$$\begin{aligned} T = T_0 &+ A_1\theta + A_2\phi + \dots \\ &+ B_1\theta' + B_2\phi' + \dots \\ &+ \tfrac{1}{2}(A_{11}\theta^2 + 2A_{12}\theta\phi + \dots) \\ &+ \tfrac{1}{2}(B_{11}\theta'^2 + 2B_{12}\theta'\phi' + \dots) \\ &+ C_{11}\theta\theta' + C_{12}\theta\phi' + C_{21}\theta'\phi + \dots \\ &+ \text{\&c.} \end{aligned}$$

In the same way we may make an expansion for the Potential Energy of the forces, viz.

$$\begin{aligned} V = E_0 &+ E_1\theta + E_2\phi + \dots \\ &+ \tfrac{1}{2}(E_{11}\theta^2 + 2E_{12}\theta\phi + \dots), \end{aligned}$$

when these two functions are given the whole dynamical system and the forces are known; and we may form the equations of motion by Lagrange's method.

2. We shall here however limit the question by supposing that the motion about which the system is oscillating is what has been called in Chap. I. *steady*. The analytical peculiarity of such a motion is that when referred to proper co-ordinates, every coefficient in each of these two series is constant, i.e. independent of t. As already explained the physical peculiarities are that the vis viva is constant throughout the steady motion and the same oscillations follow from the same disturbance at whatever instant it may be applied to the motion. A method of discovering the proper co-ordinates, if unknown, will be given a little further on.

3. In order to form the equations of motion we must now substitute in Lagrange's equations

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} + \frac{dV}{d\theta} = 0,$$
$$\text{\&c.} = 0,$$

rejecting all the squares of small quantities. The steady motion being given by θ, ϕ, &c. all zero, each of these must be satisfied when we omit the terms containing θ, ϕ, &c. We thus obtain the equations of steady motion, viz.

$$A_1 = E_1,$$
$$A_2 = E_2,$$
$$\text{\&c.} = \text{\&c.}$$

These equations may be simply formed in any case by the following rule. Putting $L = T - V$, so that L is the difference between the kinetic and potential energies, expand L in powers of the co-ordinates θ, ϕ, &c. regarding θ', ϕ', &c. as zero. The required relations are obtained by equating the coefficients of the first powers to zero. This rule may be also expressed thus. *Let* L *be the general expression for the excess of the kinetic energy over the potential energy of a dynamical system in terms of its* n *co-ordinates* ξ, η, &c. *Let this system be moving in steady motion with constant values of* $\frac{d\xi}{dt}, \frac{d\eta}{dt}$, &c.* *Then substituting these constant values in the general expression for* L, *the relations between the constants of steady motion are given by*

$$\frac{dL}{d\xi} = 0, \quad \frac{dL}{d\eta} = 0, \quad \&c.$$

In this way we obtain in general as many equations as there are co-ordinates. Usually the coefficients in the expression for T are constant because some of the co-ordinates are constant in the state of steady motion, and the other co-ordinates appear in the expressions for T and V only as differential coefficients. In such cases we have clearly fewer equations than co-ordinates to connect the constants of steady motion. We have then a system of possible steady motions which we may conveniently term *parallel steady motions.*

4. To obtain the equations to the oscillatory motion, we retain the first powers of θ, ϕ, &c. We thus obtain a series of equations of which the following is a specimen:

$$\begin{aligned}&\left(B_{11}\frac{d^2}{dt^2} - A_{11} + E_{11}\right)\theta\\ &+\left\{B_{12}\frac{d^2}{dt^2} + (C_{21} - C_{12})\frac{d}{dt} - A_{12} + E_{12}\right\}\phi\\ &+\left\{B_{13}\frac{d^2}{dt^2} + (C_{31} - C_{13})\frac{d}{dt} - A_{13} + E_{13}\right\}\psi\\ &+ \dots\dots\dots\dots = 0.\end{aligned}$$

To solve these we write

$$\theta = M_1 e^{mt}, \quad \phi = M_2 e^{mt}, \quad \psi = M_3 e^{mt}, \quad \&c.$$

Substituting we obtain on eliminating the ratios $M_1 : M_2 : M_3$ &c. a determinantal equation, viz.

* Since we may in Art. 1 change the co-ordinates from ξ, η, ζ, &c. to ξ_1, η_1, ζ_1, &c., where $\xi = f(\xi_1)$ $\eta = F(\eta_1)$ &c., it is clear that the steady motion can be always expressed by *constant* values of the differential coefficients of the co-ordinates.

$B_{11}m^2 - A_{11} + E_{11}$	$B_{12}m^2 - A_{12} + E_{12}$ $+ (C_{21} - C_{12})\, m$	$B_{13}m^2 - A_{13} + E_{13}$ $+ (C_{31} - C_{13})\, m$	&c.	
$B_{12}m^2 - A_{12} + E_{12}$ $- (C_{21} - C_{12})\, m$	$B_{22}m^2 - A_{22} + E_{22}$	$B_{23}m^2 - A_{23} + E_{23}$ $+ (C_{32} - C_{23})\, m$	&c.	$= 0.$
$B_{13}m^2 - A_{13} + E_{13}$ $- (C_{31} - C_{13})\, m$	$B_{23}m^2 - A_{23} + E_{23}$ $- (C_{32} - C_{23})\, m$	$B_{33}m^2 - A_{33} + E_{33}$	&c.	
&c.	&c.	&c.	&c.	

This equation will be referred to as the *Determinantal equation.*

5. If we refer to the equation formed by this determinant and read it in horizontal lines, we have of course the several equations of motion, each term being the coefficient of θ, ϕ, ψ, &c. in order. In this form the equations may be reproduced by the following easy rule.

Taking the expression for $T - V$ as given in Art. 1, let us consider only the terms of the second order, those of the first having been already used to determine the steady motion as explained in Art. 3. Separate from the rest, the even powers of θ, θ', ϕ, ϕ', &c. and write for θ'^2, $\theta'\phi'$, &c. $-D^2\theta^2$, $-D^2\theta\phi$, &c., so that D will stand either for $\frac{d}{dt}$ or for the m in the determinant of Art. 4, when we write $\theta = M_1 e^{mt}$, $\phi = M_2 e^{mt}$, &c. Let the sum of these terms be called P, so that

$$P = \tfrac{1}{2}(A_{11} - E_{11} - B_{11}D^2)\,\theta^2 + (A_{12} - E_{12} - B_{12}D^2)\,\theta\phi + \&c.$$

Let the remaining portion of the terms of $T - V$, viz. those containing both θ, ϕ, &c. and θ', ϕ', &c., be called Q, so that

$$Q = C_{11}\theta\theta' + C_{12}\theta\phi' + C_{21}\phi\theta' + \dots$$

Then the several equations may be formed from the rule

$$\left.\begin{aligned} \frac{dP}{d\theta} + \frac{dQ}{d\theta} - D\,\frac{dQ}{d\theta'} &= 0 \\ \frac{dP}{d\phi} + \frac{dQ}{d\phi} - D\,\frac{dQ}{d\phi'} &= 0 \\ \&c. &= 0 \end{aligned}\right\}.$$

In applying this rule no accented letters will occur except in the second term of each equation. If we wish D to stand for m

in the determinant, we must regard θ', ϕ', &c. as abbreviations for $D\theta$, $D\phi$, &c. If we wish to use the equations themselves, we replace $D\theta$, $D\phi$, &c. by θ', ϕ', &c.

[The determinant may also be found by another rule. Taking as before only the terms of the second order in the Lagrangian function $L = T - V$, let us separate the terms of the form

$$Q = C_{11}\theta\theta' + C_{12}\theta\phi' + C_{21}\theta'\phi + \&c.$$

In the remaining terms put $\theta' = \theta m\sqrt{-1}$, $\phi' = \phi m\sqrt{-1}$, and so on, and write down the *discriminant*. If the system oscillates about a position of equilibrium the terms represented by Q are absent and the discriminant thus formed will be the determinantal equation giving the required values of m. But if the system oscillate about a state of steady motion we must modify the discriminant by adding some quantity derived from Q to each term. To find this, write above the columns θ', ϕ', &c. and before the rows θ, ϕ, &c. Consider any term, say the term in the column ϕ' and row θ. We must add to that term $(C_{12} - C_{21})\,m$, where $C_{12} - C_{21}$ is the excess of the coefficient of $\phi'\theta$ above the coefficient of $\phi\theta'$ in the expression for Q. Since the determinant is unchanged by writing $-m$ for m, we may, if preferred, add the excess of the coefficient of $\theta'\phi$ above the coefficient of $\theta\phi'$ provided we adhere to one order throughout.]

6. If in the determinantal equation we write $-m$ for m, the rows of the new determinant are the same as the columns of the old, so that the determinant is unaltered. When expanded, we shall have an equation which contains only *even* powers of m.

The condition of dynamical stability is that the roots of this equation should be all of the form

$$m = \pm\beta\sqrt{-1}.$$

If we write $m^2 = -p^2$, the roots of the transformed equation must be all real.

In the case of equal roots of the form $m = \pm\beta\sqrt{-1}$, it has been shown in Art. 5 of Chap. I. that it is necessary for stability that the proper number of minors in this determinant should vanish. If there be two equal roots, these roots must make all the first minors zero; if three equal roots, all the first and second minors must vanish, and so on.

7. When the system depends on many co-ordinates the labour of expanding this determinant is often considerable. Methods of evading this in certain cases will be given in the following chap-

ters. But when the development is necessary we may proceed in the following manner. Let the determinant be written

$$\begin{vmatrix} B_{11}m^2 + a_{11} & B_{12}m^2 + a_{12} & B_{13}m^2 + a_{13} \\ & + F_{12}m & + F_{13}m \\ B_{12}m^2 + a_{12} & B_{22}m^2 + a_{22} & B_{23}m^2 + a_{23} \\ + F_{21}m & & + F_{23}m \\ \&c. & \&c. & \&c. \end{vmatrix},$$

and let

$$B = B_{11}\frac{\theta^2}{2} + B_{12}\theta\phi + \ldots,$$

$$a = a_{11}\frac{\theta^2}{2} + a_{12}\theta\phi + \ldots,$$

$$F_{pq} = -F_{qp}.$$

We know that the determinant when expanded is of an even order, hence all odd powers of m must finally vanish. Let us expand the determinant in powers of the F's. The first term is the discriminant of $Bm^2 + a$, this term is independent of the F's. The terms which contain the first powers of the F's are obtained by erasing any one line of this discriminant and replacing it by the corresponding F terms. But these terms all vanish and we need not describe them minutely. The terms containing the products and squares of the F's may be obtained by erasing every two lines of the discriminant and replacing them by the corresponding F terms. Thus if we erase the two first lines we have the determinant

$$\begin{vmatrix} 0 & F_{12}m & F_{13}m & \&c. \\ -F_{12}m & 0 & F_{23}m & \&c. \\ B_{13}m^2 + a_{13} & B_{23}m^2 + a_{23} & B_{33}m^2 + a_{33} & \&c. \\ \&c. & \&c. & \&c. & \end{vmatrix},$$

and so on for all the other rows taken two and two. The terms which contain the cubes and all odd powers of the F's vanish, while the terms which contain the fourth powers may be obtained by erasing four lines of the discriminant and replacing them by the corresponding F's.

When the determinant has been expanded, we have an equation of an *even* order to find the values of m. We may therefore employ the short method of Art. 5, Chap. III., to obtain the Sturmian functions.

8. Necessary and sufficient tests of the stability of the motion of a system of bodies are given in the preceding pages. But it

is assumed, as explained in Art. 2, that the co-ordinates have been properly chosen. They are supposed to have been so chosen that the coefficients in the expanded Lagrangian function are all constants. When this is not the case we must discover the proper co-ordinates to which the system must be referred before we can apply the test of stability. But when the motion is steady this is not difficult.

There are obviously many such systems of co-ordinates, and one set may generally be found by a simple examination of the steady motion. If there are any quantities which are constant during the steady motion, they will often serve for some of the co-ordinates. Others may be found by considering what quantities appear only as differential coefficients or velocities. Practically these will be the most convenient methods of discovering proper co-ordinates, since no further change will then be necessary and we may at once form the determinant of stability. But if these methods fail we may adopt the following analytical method of transforming (where possible) the general Lagrangian function with variable coefficients into one with constant coefficients.

9. Let the Lagrangian function be

$$\begin{aligned} L = L_0 &+ A_1\theta + A_2\phi + \&c. + B_1\theta' + B_2\phi' + \&c. \\ &+ \tfrac{1}{2} A_{11}\theta^2 + A_{12}\theta\phi + \dots \\ &+ \tfrac{1}{2} B_{11}\theta'^2 + B_{12}\theta'\phi' + \dots \\ &+ C_{11}\theta\theta' + C_{12}\theta\phi' + C_{21}\phi\theta' + \dots \end{aligned}$$

where the coefficients are all functions of t and the co-ordinates θ, ϕ, &c. have been so chosen as to vanish along the steady motion. We have therefore for the steady motion

$$\frac{d}{dt}B_1 - A_1 = 0,$$
$$\&c. = 0.$$

The oscillations about the steady motion are given by the terms of the second order. Our present object is to transform these to others with constant coefficients by the following substitutions:

$$\begin{aligned} \theta &= p_1x + p_2y + p_3z + \&c., \\ \phi &= q_1x + q_2y + q_3z + \&c., \\ \&c. &= \&c., \end{aligned}$$

where the p's, q's, &c. are functions of the time at our disposal.

Substituting and equating the coefficients of x'^2, y'^2, &c. to unity, we have as many equations of the form

$$\tfrac{1}{2} B_{11}p^2 + \tfrac{1}{2} B_{22}q^2 + B_{12}pq + \dots = 1 \text{ .... } (1)$$

as there are co-ordinates. Equating the coefficients of the products $x'y'$, $x'z'$, &c. to zero we get $n\frac{n-1}{2}$ equations of the form

$$B_{11}p_1p_2 + B_{22}q_1q_2 + B_{12}(p_1q_2 + p_2q_1) + \ldots = 0 \ldots\ldots\ldots (2),$$

supposing that there are n co-ordinates.

Equating to the constants α_1, α_2, ... the coefficients of xx', yy', &c. having subtracted the differential coefficients of (1) we have n equations of the form

$$(C_{11} - \tfrac{1}{2}B_{11}')p^2 + (C_{22} - \tfrac{1}{2}B_{22}')q^2 + (C_{12} + C_{21} - B_{12}')pq + \ldots = \alpha \ldots (3).$$

Adding the coefficients of xy' and $x'y$ and subtracting the differential coefficients of (2) we have $n\frac{n-1}{2}$ equations of the form

$$\left.\begin{array}{l}(2C_{11} - B_{11}')p_1p_2 + (2C_{22} - B_{22}')q_1q_2 \\ + (C_{12} + C_{21} - B_{12}')(p_1q_2 + p_2q_1) + \ldots\end{array}\right\} = 0 \ldots\ldots (4).$$

Equations (1), (2) and (4) give n^2 equations to find the n^2 quantities p_1p_2, &c., q_1q_2, &c. The solution of these equations is a purely geometrical problem. If we construct the two quadrics

$$\tfrac{1}{2}B_{11}\theta^2 + \tfrac{1}{2}B_{22}\phi^2 + B_{12}\theta\phi + \ldots = 1,$$

$$(C_{11} - \tfrac{1}{2}B'_{11})\theta^2 + (C_{22} - \tfrac{1}{2}B'_{22})\phi^2 + (C_{12} + C_{21} - B'_{12})\theta\phi + \ldots = 1,$$

and refer them to their common conjugate diameters, by writing

$$\theta = p_1x + p_2y + \ldots$$
$$\phi = q_1x + q_2y + \ldots$$
$$\text{\&c.} = \text{\&c.},$$

making the first quadric to become what we may call a sphere by projection; the values of p_1p_2, &c., q_1q_2, &c. thus found are the values required to make some of the coefficients in the Lagrangian function become constant. These values must of course make all the other coefficients of the second order in the Lagrangian function constants also, and thus we have $n(n+1)$ analytical conditions that the motion should be steady.

It might be supposed that greater generality would be obtained by replacing the zero's of equations (2) and (4) or the unities of (1) by arbitrary constants. This may be convenient in practice, but as we know that by a subsequent real change with constant values of p_1p_2, &c., q_1q_2, &c., we can render them zero or unity, it simplifies the argument to perform the two transformations at once.

10. The geometrical problem just alluded to admits of a real solution whenever one quadric can be projected by a real projection into a sphere. The problem then becomes that of finding the principal axes of the other. This is just our case, since the expression

$$\tfrac{1}{2}B_1\,\theta'^2+\beta_{12}\theta'\phi'+\dots$$

is necessarily positive for all values of $\theta'\phi'$.

It is unnecessary to describe here the mode of solving this problem. It is sufficient to say that it may be reduced to the solution of the symmetrical determinantal equation

$$\begin{vmatrix} B_{11}-\lambda D_{11}, & D_{12}-\lambda D_{12} & \dots \\ B_{12}-\lambda D_{12}, & D_{22}-\lambda D_{22} & \dots \\ \dots\dots & \dots\dots & \end{vmatrix}=0,$$

where D_{11}, D_{12}, &c. are the coefficients of θ^2, $\theta\phi$, &c. in the second of the quadrics. The roots of this equation are known to be real when the suppositions just mentioned are satisfied.

11. In order to examine the fundamental determinant in Art. 4 a little more closely, let us suppose it reduced to depend on three co-ordinates. We may then have the advantage of a geometrical analogy. Let the co-ordinates be ξ η, ζ, and let the equations of motion be written

$$\left(B_{11}\frac{d^2}{dt^2}-A_{11}\right)\xi+\begin{pmatrix}B_{12}\dfrac{d^2}{dt^2}-A_{12}\\ -G\dfrac{d}{dt}\end{pmatrix}\eta+\begin{pmatrix}B_{13}\dfrac{d^2}{dt^2}-A_{13}\\ +F\dfrac{d}{dt}\end{pmatrix}\zeta=0,$$

$$\begin{pmatrix}B_{12}\dfrac{d^2}{dt^2}-A_{12}\\ +G\dfrac{d}{dt}\end{pmatrix}\xi+\left(B_{22}\frac{d^2}{dt^2}-A_{22}\right)\eta+\begin{pmatrix}B_{23}\dfrac{d^2}{dt^2}-A_{23}\\ -E\dfrac{d}{dt}\end{pmatrix}\zeta=0,$$

$$\begin{pmatrix}B_{13}\dfrac{d^2}{dt^2}-A_{13}\\ -F\dfrac{d}{dt}\end{pmatrix}\xi+\begin{pmatrix}B_{23}\dfrac{d^2}{dt^2}-A_{23}\\ +E\dfrac{d}{dt}\end{pmatrix}\eta+\left(B_{33}\frac{d^2}{dt^2}-A_{33}\right)\zeta=0.$$

Let a geometrical point P move in space so that its co-ordinates referred to any axes are ξ, η, ζ. Then the position and motion of the point exactly give us the position and motion of the system.

12. Looking at the equations of motion just written down we see that they are similar to those which give the oscillations of a

system about a position of equilibrium, but that there are in addition terms $E\frac{d\eta}{dt}$, $F\frac{d\zeta}{dt}$, &c. The general effect of these terms, as will appear from what follows in the subsequent chapter, is to increase the stability. If we transpose these terms to the other sides of the equations, we may regard them as impressed forces acting on the system, whose resolved parts in the directions of the axes ξ, η, ζ, are

$$\left.\begin{aligned} X &= G\frac{d\eta}{dt} - F\frac{d\zeta}{dt} \\ Y &= E\frac{d\zeta}{dt} - G\frac{d\xi}{dt} \\ Z &= F\frac{d\xi}{dt} - E\frac{d\eta}{dt} \end{aligned}\right\}.$$

We see at once that

$$\left.\begin{aligned} EX + FY + GZ = 0 \\ \frac{d\xi}{dt}X + \frac{d\eta}{dt}Y + \frac{d\zeta}{dt}Z = 0 \end{aligned}\right\},$$

so that these forces are at once orthogonal to the path of the representative point P and also orthogonal to the straight line whose direction cosines are proportional to E, F, G. These forces are therefore of the nature of *centrifugal forces*, as if they were produced by the rotation of the system about this straight line.

13. We may show that the straight line (E, F, G) is fixed in space. To prove this, let us transform our co-ordinates from ξ, η, ζ to x, y, z, where x, y, z are connected with ξ, η, ζ by any linear relations, such as

$$\left.\begin{aligned} \xi &= a_1x + b_1y + c_1z \\ \eta &= a_2x + b_2y + c_2z \\ \zeta &= a_3x + b_3y + c_3z \end{aligned}\right\}.$$

Let the portion of the Lagrangian function under consideration (Art. 1) be

$$C_{11}\xi\xi' + C_{12}\xi\eta' + C_{21}\eta\xi' + \ldots$$

then $\qquad G = C_{12} - C_{21},\ E = C_{23} - C_{32},\ F = C_{31} - C_{13}.$

Substituting for ξ, η, ζ their values in terms of x, y, z, we find that the difference between the coefficients of xy' and $x'y$ is

$$G' = G\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + E\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + F\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix},$$

with similar equations for E' and F'. But if μ be the determinant of transformation

$$\mu z = \zeta \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \xi \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \eta \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix},$$

with similar equations for x and y. The *ratios* of E, F, G are therefore transformed as if they were co-ordinates. If the transformation be a real transformation of Cartesian co-ordinates, let lengths each equal to unity be measured from the origin along the axes $O\xi$, $O\eta$, $O\zeta$, thus forming a tetrahedron whose volume is V. Let a similar construction be made for the new axes, forming a tetrahedron of volume V'. Then* $\mu = \frac{V'}{V}$. Hence the quantities $\frac{E}{V}, \frac{F}{V}, \frac{G}{V}$ may be transformed as if they were lengths measured along the axes and become $\frac{E'}{V'}, \frac{F'}{V'}, \frac{G'}{V'}$. If both systems of co-ordinates are rectangular we have $V = \frac{1}{3}$, $V' = \frac{1}{3}$,

Let ω be the resultant of $\frac{E}{V}, \frac{F}{V}, \frac{G}{V}$, then ω may be regarded as a fixed length measured from the origin along a straight line fixed in space.

Let v be the velocity of the representative particle, θ the angle between the direction of this velocity and the axis whose direction cosines are proportional to E, F, G. Then the resultant of the forces X, Y, Z is easily seen to be $2vV\omega \sin\theta$ acting perpendicular to the axis and to the direction of the motion. We might call the straight line (EFG) the axis of the centrifugal forces.

[* Let $(\xi_1 \eta_1 \zeta_1)$, $(\xi_2 \eta_2 \zeta_2)$, $(\xi_3 \eta_3 \zeta_3)$ be the co-ordinates of three points A, B, C referred to any *oblique* co-ordinates. Let us find the volume V' of the tetrahedron of which these and the origin are the angular points. Since the volume vanishes when any angular point as C lies in the plane containing the origin and the other two A, B, the expression for the volume must contain the factor

$$\mu = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix}.$$

The volume is evidently an integral rational function of the co-ordinates when the axes are rectangular and the plane AOB is taken as the plane of xy, it easily follows that this is true for all axes. Since this function cannot be of more than the third order, we have $V' = M\mu$, where M is independent of the co-ordinates of A, B, C. When the points A, B, C are on the axes at unit distances from the origin, let V be the volume of the tetrahedron. In this case $\mu = 1$, and $\therefore M = V$. We have therefore in all cases $V' = V\mu$.

In the text, let the extremities of the unit lengths measured along the axes of x, y, z be called A, B, C. Then the $(\xi \eta \zeta)$ co-ordinates of A, B, C are (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), respectively. Hence by what has just been said $V' = V\mu$.]

14. The expressions for the co-ordinates in terms of the time will in general contain as many periodic functions as there are co-ordinates. If the initial conditions are such that each contains one and the same periodic function, the motion recurs after a constant interval and the system is said to be performing a *simple* or *harmonic oscillation*.

If the system be oscillating about a position of equilibrium, with a Lagrangian function

$$A_{11}\xi^2 + 2A_{12}\xi\eta + \dots$$
$$+ B_{11}\xi'^2 + 2B_{12}\xi'\eta' + \dots$$

we know* that the harmonic oscillations are represented by *rectilinear* motions of the representative particle, and that these are along the common conjugate diameters of the two quadrics ... equations are

$$\left.\begin{aligned} A_{11}\frac{\xi^2}{2} + A_{12}\xi\eta + \dots &= a \\ B_{11}\frac{\xi^2}{2} + B_{12}\xi\eta + \dots &= b \end{aligned}\right\},$$

where a and b are two constants chosen to make the quadrics real. Let us consider what are the harmonic paths of the representative point when the system is oscillating about a state of steady motion.

In any harmonic vibration we have $\xi = L \cos(\lambda t + a)$ with similar equations for η and ζ. Hence

$$\frac{d^2\xi}{dt^2} = -\lambda^2\xi, \quad \frac{d^2\eta}{dt^2} = -\lambda^2\eta, \quad \frac{d^2\zeta}{dt^2} = -\lambda^2\zeta.$$

Substitute these in the equations of Art. 11. Differentiate and substitute again. Multiply by ξ, η, ζ, add and integrate, we obtain

$$\left(B_{11}\frac{\xi^2}{2} + B_{12}\xi\eta + \dots\right)\lambda^2 + \left(A_{11}\frac{\xi^2}{2} + A_{12}\xi\eta + \dots\right) = c,$$

where c is some constant. The harmonic path lies on this quadric, which has a common set of conjugate diameters with the two quadrics a and b.

If we resume the result of the substitution of $\frac{d^2\xi}{dt^2}$, &c. in the equations of Art. 11, and multiply by E, F, G respectively and add, we obtain

$$[(B_{11}E + B_{12}F + B_{13}G)\,\xi + \&c.]\,\lambda^2 + [(A_{11}E + \dots)\,\xi + \&c.] = 0,$$

* A short paragraph in Thomson and Tait's *Natural Philosophy*, page 273, is the only notice of this which the author has discovered.

which is a plane, and is diametral to the straight line (EFG) with regard to the quadric c.

The harmonic paths are therefore ellipses. The three harmonic planes are diametral to the same straight line and this straight line is fixed in space, being the axis of the centrifugal forces.

If we eliminate λ between the equations to the plane and the quadric c, we get a cubic surface on which the three harmonic conics lie.

If E, F, G are zero, which is the case when the system oscillates about a position of equilibrium, the quadric c becomes a cylinder. This may be conveniently shown by referring the system to such co-ordinates that the coefficients B_{12}, B_{13}, B_{23}, A_{12}, A_{13}, A_{23} are all zero. In this case the diametral plane of every straight line passes through the axis of the cylinder. The harmonic oscillations are therefore rectilinear.

If R be the length of that semidiameter of the quadric (c) which is parallel to the fixed straight line (E, F, G), it may be shown that the

$$\left.\begin{array}{r}\text{Product of the axes}\\ \text{of the quadric } c\end{array}\right\} = \frac{R}{\sqrt{E^2+F^2+G^2}} \cdot \frac{2c}{\lambda}.$$

If E, F, G are all zero, and their ratio is indeterminate, R is any diameter. Hence one of the axes of the quadric (c) must be infinite and the quadric will be a cylinder.

[If the quadric (c) be a cylinder and E, F, G are not all zero, we must have either λ zero or R infinite. In the latter case the axis of the cylinder will coincide with the axis of the centrifugal forces.]

The quadric (c) has also the following geometrical property. Let the lengths of semidiameters of the quadrics (a) and (b) drawn parallel to the axis of the centrifugal forces be ρ and ρ'. Through the intersection of these quadrics describe a quadric so that the

$$\left.\begin{array}{r}\text{Product of}\\ \text{its axes}\end{array}\right\} = \frac{2\sqrt{ab}}{\sqrt{E^2+F^2+G^2}} \cdot \frac{\left(\dfrac{1}{\rho'^2}-\dfrac{1}{\rho^2}\right)R}{\sqrt{\left(\dfrac{1}{\rho'^2}-\dfrac{1}{R^2}\right)\left(\dfrac{1}{R^2}-\dfrac{1}{\rho^2}\right)}}.$$

This quadric is similar to the quadric (c).

15. The introduction of the representative point to exhibit the motion of a system may appear somewhat artificial. If however we properly choose the co-ordinates the particle moves exactly as a free particle, and we might reduce the problem of finding the

oscillations of a system to a problem in Dynamics of a particle. Refer the quadric

$$B_{11}\frac{\xi^2}{2} + B_{12}\xi\eta + \ldots = b$$

to its principal axes and let the equation thus changed be

$$B_{11}'\frac{\xi_1^2}{2} + B_{22}'\frac{\eta_1^2}{2} + \ldots = b.$$

Since B_{11}', B_{22}', &c. are positive quantities, we may put

$$\sqrt{B_{11}'}\xi_1 = x, \quad \sqrt{B_{22}'}\eta_1 = y, \quad \&c.$$

The quadric has thus been "projected" into a sphere. Let x, y, z be now chosen as the co-ordinates of the system and let the Lagrangian function be expressed in the form

$$L = x'^2 + y'^2 + z'^2 + \tfrac{1}{2} A_{11}x^2 + A_{12}xy + \ldots$$
$$+ C_{11}xx' + \ldots$$

the terms of the first degree being omitted as not necessary to our present purpose. The three equations of motion at the beginning of Art. 11 take the form

$$\left(\frac{d^2}{dt^2} - A_{11}\right)x + \left(-A_{12} - G\frac{d}{dt}\right)y + \left(-A_{13} + F\frac{d}{dt}\right)z = 0,$$
$$\&c. = 0,$$

which are the three equations of motion of a free single particle whose co-ordinates are x, y, z under the action of forces whose force function is

$$\tfrac{1}{2}A_{11}x^2 + A_{12}xy + \ldots$$

and a force acting perpendicular to the path and also perpendicular to a fixed straight line, the force being proportional to the velocity.

16. As an illustration of this theory, let us here make a short digression. However the particles of light may oscillate, whether in a rotatory or linear manner, we know the motion is related to a certain plane called the plane of polarization. It may be shown that any harmonic oscillation about a position of *equilibrium* may be represented by a *rectilinear* oscillation of the representative particle. Let us represent the motion at any point of the ether by a rectilinear oscillation in a direction perpendicular to the plane of polarization. This would be Fresnel's Vibration. The representative particle, as just shown, would not necessarily move as if it were a free single particle. But let us assume (and a proof is not necessary to our present purpose) that when the oscillation is drawn as above described the motion in the plane of the front

is the same as that of a free particle, while that perpendicular is not free. On this assumption we see that Fresnel in his theory of double refraction is justified in taking actual instead of relative displacements, for it is the representative particle he is considering. He also neglects the force normal to the front, for the particle moves as a free particle only in the plane of the front. These general remarks are not meant to explain Fresnel's theory, but merely to show how the representative particle may be used to replace a complicated motion.

17. [When a system is performing a harmonic oscillation about a state of steady motion or about a position of equilibrium, the motion repeats itself continually at a constant period, that is to say, the values of the co-ordinates recur at this interval. This is the chief peculiarity of a harmonic oscillation. When the oscillation is about a position of equilibrium, the representative particle oscillates in a straight line whose middle point represents the position of equilibrium. Thus the system passes through the position of equilibrium twice in each complete oscillation. When the oscillation is about a state of steady motion the path of the representative particle is an ellipse whose centre is at the point occupied by the system in steady motion at the same instant. Thus the system does not in general ever coincide with the simultaneous position of the system in the undisturbed or steady motion. When a system is disturbed by a small impulse from a state of steady motion, it will in general describe a compound oscillation made up of at least two harmonic oscillations, at the instant of disturbance these two neutralize each other so that in the disturbed and undisturbed motions two simultaneous positions are coincident. But it is clear that this cannot occur again unless either the periods of the two harmonics are commensurable or the period of one of them is infinite.]

18. [In some cases the ellipse degenerates into a straight line. Thus if the quadric (*c*) be a cylinder the diametral plane of the axis of the centrifugal forces will pass through the axis of the cylinder, and thus the harmonic oscillation corresponding to this particular value of λ will be rectilinear. In this case the system twice in each oscillation passes through the position it would have occupied at the same instant in the undisturbed motion.

The quadric (*c*) has a common set of conjugate diameters with the quadrics (*a*) and (*b*). Hence if (*c*) be a cylinder, its axis must be parallel to one of the three common conjugate diameters of (*a*) and (*b*). If we refer the quadrics (*a*) and (*b*) to their common conjugate diameters, they take the form

$$\left.\begin{aligned} A_{11}'\xi^2 + A_{22}'\eta^2 + A_{33}'\zeta^2 &= 2a \\ B_{11}'\xi^2 + B_{22}'\eta^2 + B_{33}'\zeta^2 &= 2b \end{aligned}\right\}.$$

The cylinder which passes through their intersection and has its axis parallel to the diameter ζ is found by eliminating ζ^2 between these equations. We see therefore that $B_{33}'\lambda^2 + A_{33}' = 0$. If then the axis of the cylinder cut the quadrics (a) and (b) in D and D' respectively, we find that for this oscillation

$$\lambda^2 = \frac{OD'^2}{OD^2} \cdot \frac{-a}{b}.$$

It has already been shown that when this value is finite, the direction of ODD' is along the axis of the centrifugal forces.]

19. [In some cases two or more of the values of λ are zero. In these cases the co-ordinates will have terms of the form $nt + \epsilon$, where n and ϵ are two small constants. When, as explained in Art. 3 of this Chapter, there are several *parallel* states of steady motion, these terms imply that the motion is stable about a state of steady motion very nearly the same as the undisturbed motion but not coincident with it. The actual undisturbed motion, unless n is zero, is unstable in the sense that if a proper disturbance be given to the system, the system will depart widely from the positions it would have simultaneously occupied in the undisturbed motion.]

20. In many cases of small oscillations it will be found that the Lagrangian function $T - V$ is not a function of some of the co-ordinates as θ, ϕ, &c. though it is a function of their differential coefficients θ', ϕ', &c. In such cases the steady motion will be usually given by constant values of these differential coefficients, while the other co-ordinates as ξ, η, &c. are also constant. It is evident that the determinantal equation of Art. 4 is needlessly complicated. It is clear that there will be as many pairs of roots equal to zero as there are co-ordinates θ, ϕ, &c. It will be an advantage to eliminate θ', ϕ', &c. altogether from the Lagrangian function, and to find the remaining roots by operating only with the co-ordinates ξ, η, &c. We shall thus obtain a determinant with just as many rows as there are co-ordinates of the kind ξ, η, &c.

Let L_1 be the Lagrangian function expressed as a function of $\theta'\phi'$, &c. $\xi\eta$, $\xi'\eta'$, &c. Let L_2 be its value when $\theta'\phi'$ are eliminated, so that L_2 is a function of $\xi\eta$, $\xi'\eta'$, &c. only. To effect this elimination we have the integrals

$$\frac{dT}{d\theta'} = c_1, \quad \frac{dT}{d\phi'} = c_2, \text{ \&c.}$$

where c_1, c_2, &c. are constants. Then

$$\frac{dL_2}{d\xi} = \frac{dL_1}{d\xi} + \frac{dL_1}{d\theta'} \cdot \frac{d\theta'}{d\xi} + \frac{dL_1}{d\phi'} \cdot \frac{d\phi'}{d\xi} + \text{\&c.}$$

$$= \frac{dL_1}{d\xi} + c_1 \frac{d\theta'}{d\xi} + c_2 \frac{d\phi'}{d\xi} + \dots$$

$$\frac{dL_2}{d\xi} = \frac{dL_1}{d\xi} + c_1 \frac{d\theta'}{d\xi} + c_2 \frac{d\phi'}{d\xi} + \dots$$

$$\therefore \frac{d}{dt}\frac{dL_2}{d\xi'} - \frac{dL_2}{d\xi} = \frac{d}{dt}\frac{dL_1}{d\xi'} - \frac{dL_1}{d\xi} + c_1\left(\frac{d}{dt}\frac{d\theta'}{d\xi'} - \frac{d\theta}{d\xi}\right) + \text{\&c.}$$

But
$$\frac{d}{dt}\frac{dL_1}{d\xi'} - \frac{dL_1}{d\xi} = 0.$$

Hence if we take
$$L' = L - c_1\theta' - c_2\phi' - \text{\&c.}$$
and eliminate θ', ϕ' *by help of the integrals* $\frac{dT}{d\theta'} = c_1$, $\frac{dT}{d\phi'} = c_2$, *we may treat* L′ *just as we do the Lagrangian function* L. *The equations giving the small oscillations about the steady motion will be*
$$\frac{d}{dt}\frac{dL'}{d\xi'} - \frac{dL'}{d\xi} = 0, \text{ \&c.} = 0.$$

The function L' may be called the *modified Lagrangian function.*

[It should be noticed that this is equivalent to a partial use of Hamilton's transformation of Lagrange's equations. Sir W. R. Hamilton eliminates *all* the differential coefficients θ', ϕ', &c. by the help of equations of the form $\frac{dT}{d\theta'} = u$, $\frac{dT}{d\phi'} = v$, &c. where u, v, &c. are made to be new variables*. In our transformation only

* The Hamiltonian transformation of Lagrange's equations bears a remarkable analogy to the transformation of Reciprocation in Geometry. This may be shown in the following manner.

When the system has three co-ordinates θ, ϕ, ψ, we may regard θ', ϕ', ψ' as the Cartesian co-ordinates of a representative point P. The position and path of P will exhibit to the eye and will determine the motion of the system. Let u, v, w be the Hamiltonian variables, so that
$$u = \frac{dT_1}{d\theta'}, \quad v = \frac{dT_1}{d\phi'}, \quad w = \frac{dT_1}{d\psi'},$$
where T_1 is the semi vis viva expressed as a function of $\theta, \phi, \psi, \theta', \phi', \psi'$. Then u, v, w may be regarded as the co-ordinates of another point Q whose position and path will also determine the motion of the system.

If the semi vis viva be given by the general expression
$$T_1 = \tfrac{1}{2} A_{11}\theta'^2 + A_{12}\theta'\phi' + \dots$$
it is clear that the point P always lies on the quadric $T_1 = U$ where U is the force function and the co-ordinates θ, ϕ, ψ have their instantaneous values. The point Q must therefore lie on another quadric which is the polar reciprocal of the first with regard to a sphere whose centre is at the origin and whose radius is equal to $\sqrt{2U}$. The equation to the reciprocal quadric is therefore
$$T_2 = -\frac{1}{2\Delta}\begin{vmatrix} 0 & u & v & w \\ u & A_{11} & A_{12} & A_{13} \\ v & A_{12} & A_{22} & A_{23} \\ w & A_{13} & A_{23} & A_{33} \end{vmatrix} = U,$$

those new variables are introduced which would be constants in Sir W. R. Hamilton's transformation.

This remark suggests an extension of the process. If L be a function of θ, ϕ, &c. as well as of θ', ϕ', &c. the quantities c_1, c_2, &c. will not be constants. We express this by writing u, v, &c. instead of c_1, c_2, &c. Suppose we wish to eliminate *some* of the differential coefficients, viz. θ', ϕ', &c. and to retain the remaining ones, viz. ξ', η', &c. If we put

$$L' = L - u\theta' - v\phi' - \&c.$$

we may easily show as in the preceding page that

$$\frac{d}{dt}\frac{dL'}{d\xi'} - \frac{dL'}{d\xi} = 0, \ \&c. = 0.$$

where Δ is the determinant, called the discriminant, which may be formed from the determinant just written down by omitting the first row and the first column.

This is a general expression for the Hamiltonian function and agrees with that which may be deduced from the result in Art. 21, when *all* the variables are transformed by the Hamiltonian process.

Since the polar reciprocal of the polar reciprocal is the original quadric, it follows that

$$\theta' = \frac{dT_2}{du}, \quad \phi' = \frac{dT_2}{dv}, \quad \psi' = \frac{dT_2}{dw},$$

which are three of the six Hamiltonian equations.

We may also show geometrically that if the coefficients of T_1 be functions of any quantity θ, then $\frac{dT_1}{d\theta} = -\frac{dT_2}{d\theta}$. To prove this we notice that if x, y, z be the co-ordinates of a point P, situated on a radius vector OP' of a quadric $\phi\,(x, y, z) = 1$ referred to its centre O, then $\phi\,(x, y, z) = \left(\frac{OP}{OP'}\right)^2$. The quadrics $T_1 = 1$ and $T_2 = 1$ may be regarded as polar reciprocals of each other with regard to a sphere whose radius is $\sqrt{2}$ and whose centre is the common centre of the two quadrics. Let P be any point on the quadric $T_1 = 1$, and let the radius vector be produced to Z so that $OP \,.\, OZ = 2$, then the quadric $T_2 = 1$ touches a plane drawn through Z perpendicular to OP and Q is the point of contact. Let these quadrics be slightly altered in consequence of a variation of θ, so that their equations are now $T_1 + dT_1 = 1$ and $T_2 + dT_2 = 1$. Let OP and OQ produced cut these quadrics respectively in P' and Q'. Then

$$T_1 + dT_1 = \left(\frac{OP}{OP'}\right)^2, \quad T_2 + dT_2 = \left(\frac{OQ}{OQ'}\right)^2.$$

Now if Z' be a point on OP produced so that $OP' \,.\, OZ' = OP \,.\, OZ$, the quadric $T_2 + dT_2$ will touch the plane drawn through Z' in some point q near Q'. The point Q' will therefore lie very nearly in the tangent plane, so that by similar triangles

$$\frac{OQ}{OQ'} = \frac{OZ}{OZ'} = \frac{OP'}{OP}.$$

Since each of these ratios is indefinitely nearly equal to unity, it follows that $dT_1 = -dT_2$.

If we put $L = T_1 + U$ and $H = T_2 - U$, Lagrange's equations may be written in the forms $u' = \frac{dL}{d\theta}$, $v' = \frac{dL}{d\phi}$, $w' = \frac{dL}{d\psi}$. Hence we have

$$-u' = \frac{dH}{d\theta}, \quad -v' = \frac{dH}{d\phi}, \quad -w' = \frac{dH}{d\psi},$$

which are the remaining three of the Hamiltonian equations.

We have thus as many equations of the Lagrangian form as there are variables ξ, η, &c. Also since $u = \frac{dL}{d\theta'}$, &c. we have by differentiation

$$\frac{dL'}{du} = \left(\frac{dL}{d\theta'} - u\right)\frac{d\theta'}{du} - \theta' + \&c. = -\theta',$$

with similar equations for ϕ', &c. By Lagrange's equations we obtain

$$\frac{dL'}{d\theta} = u', \ \&c.$$

Thus we have as many sets of equations of the Hamiltonian form as there are variables θ, ϕ, &c.]

21. We may effect this elimination once for all and find a definite expression for L'.

Let the kinetic energy be

$$T = T_{\theta\theta}\frac{\theta'^2}{2} + T_{\theta\phi}\theta'\phi' + \ldots$$

Then the integrals used will be

$$\begin{aligned} T_{\theta\theta}\theta' + T_{\theta\phi}\phi' + \ldots &= c_1 - T_{\theta\xi}\xi' - T_{\theta\eta}\eta' - \ldots \\ T_{\theta\phi}\theta' + T_{\phi\phi}\phi' + \ldots &= c_2 - T_{\phi\xi}\xi' - T_{\phi\eta}\eta' - \ldots \\ \&c. &= \&c. \end{aligned}$$

For the sake of brevity let us call the right-hand members of these equations $c_1 - X$, $c_2 - Y$, &c. Since T is a homogeneous function, we have

$$\left.\begin{aligned} T = T_{\xi\xi}\frac{\xi'^2}{2} + T_{\xi\eta}\xi'\eta' + \ldots \\ + \tfrac{1}{2}\theta'(c_1 + X) + \tfrac{1}{2}\phi'(c_2 + Y) + \&c. \end{aligned}\right\},$$

$$\left.\begin{aligned} \therefore\ L' = T_{\xi\xi}\frac{\xi'^2}{2} + T_{\xi\eta}\xi'\eta' + \&c. - V \\ - \tfrac{1}{2}\theta'(c_1 - X) - \tfrac{1}{2}\phi'(c_2 - Y) - \&c. \end{aligned}\right\}.$$

If we substitute in the second line the values of θ', ϕ', &c. found by solving the integrals just written down, we have

$$L' = T_{\xi\xi}\frac{\xi'^2}{2} + T_{\xi\eta}\xi'\eta' + \&c. - V$$

$$+\frac{1}{2\Delta}\begin{vmatrix} 0 & c_1 - X & c_2 - Y & \ldots \\ c_1 - X & T_{\theta\theta} & T_{\theta\phi} & \ldots \\ c_2 - Y & T_{\theta\phi} & T_{\phi\phi} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

where Δ is the discriminant of the terms in T, which contain only θ', ϕ', &c., and may be derived from the determinant just written down by omitting the first row and the first column.

We may expand this determinant and write it in the form

$$L' = T_{\xi\xi}\frac{\xi'^2}{2} + T_{\xi\eta}\xi'\eta' + \&c. - V$$

$$+\frac{1}{2\Delta}\begin{vmatrix} 0 & c_1 & c_2 & \dots \\ c_1 & T_{\theta\theta} & T_{\theta\phi} & \dots \\ c_2 & T_{\theta\phi} & T_{\phi\phi} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} + \frac{1}{2\Delta}\begin{vmatrix} 0 & X & Y & \dots \\ X & T_{\theta\theta} & T_{\theta\phi} & \dots \\ Y & T_{\theta\phi} & T_{\phi\phi} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$-\frac{1}{\Delta}\begin{vmatrix} 0 & X & Y & \dots \\ c_1 & T_{\theta\theta} & T_{\theta\phi} & \dots \\ c_2 & T_{\theta\phi} & T_{\phi\phi} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

where X, Y, &c. stand for

$$\left.\begin{aligned} X &= T_{\theta\xi}\xi' + T_{\theta\eta}\eta' + \dots \\ Y &= T_{\phi\xi}\xi' + T_{\phi\eta}\eta' + \dots \\ \&c. &= \&c. \end{aligned}\right\}.$$

The first of these three determinants will contain only the constants c_1, c_2, &c., and the co-ordinates ξ, η, &c. The second will not contain c_1, c_1, &c. but will be a quadratic function of ξ', η', &c. The last determinant will contain terms of the form ξ', η' with variable coefficients which may also be functions of c_1, c_2...

Since $\xi\xi'$, $\eta\eta'$, &c. are all small quantities, it is clear that this expression for L' when expanded will take a form precisely similar to that given in Art. 2, only that we have fewer variables to deal with.

22. As an example, let us consider the following problem.

A body has a point O *which is in one of the principal axes at the centre of gravity* G *fixed in space. The body is in steady motion rotating with angular velocity* n *about* OG *which is vertical. Find the conditions that the motion may be stable.*

Let OA, OB, OC be the principal axes at O and let OC coincide with the vertical OZ in steady motion. Let ξ, η be the direction cosines of the vertical OZ referred to OA, OB. Let ω_1, ω_2, ω_3 be the angular velocities about the principal axes at O. Then to the first order

$$\left.\begin{aligned} \omega_1 &= \eta' + \omega_3\xi \\ \omega_2 &= -\xi' + \omega_3\eta \end{aligned}\right\}.$$

Let θ be the angle ZOC, ψ the angle the plane ZOC makes with a plane ZOX fixed in space and ϕ the angle it makes with the plane AOC fixed in the body. Then

$$\left.\begin{array}{l}\xi = -\sin\theta\cos\phi \\ \eta = \quad \sin\theta\sin\phi\end{array}\right\}.$$

$$\omega_3 = \phi' + \psi'\cos\theta$$

$$= \phi' + (\chi' - \phi')\left(1 - \frac{\theta^2}{2}\right),$$

putting $\chi = \phi + \psi$. We easily find

$$\omega_3 = \chi' - \chi'\frac{\xi^2+\eta^2}{2} - \tfrac{1}{2}(\xi\eta' - \xi'\eta).$$

If then A, B, C be the principal moments of inertia at O, the Lagrangian function is

$$L = \frac{C}{2}\left\{\chi'\left(1 - \frac{\xi^2+\eta^2}{2}\right) - \tfrac{1}{2}(\xi\eta' - \xi'\eta)\right\}^2$$

$$+ \frac{A}{2}(\eta' + \chi'\xi)^2 + \frac{B}{2}(-\xi' + \chi'\eta)^2$$

$$- Mgh\left(1 - \frac{\xi^2+\eta^2}{2}\right),$$

where M is the mass of the body, and $h = OG$.

Since χ is absent from the equation we have the integral

$$\frac{dL}{d\chi'} = c_1,$$

which gives

$$\chi' = n + \text{terms of second order.}$$

Hence

$$L' = L - Cn\chi'$$

$$= \text{const} + \frac{A}{2}\eta'^2 + \frac{B}{2}\xi'^2$$

$$+ \{(A-C)n^2 + Mgh\}\frac{\xi^2}{2} + \{(B-C)n^2 + Mgh\}\frac{\eta^2}{2}$$

$$+ \left(A - \frac{C}{2}\right)n\xi\eta' - \left(B - \frac{C}{2}\right)n\xi'\eta.$$

Using this as the Lagrangian function we easily find

$$\begin{vmatrix}(A-C)n^2 + Mgh - Bm^2, & (A+B-C)\,nm \\ -(A+B-C)\,nm, & (B-C)n^2 + Mgh - Am^2\end{vmatrix} = 0.$$

The roots of this equation to find m must for stability be of the form $\pm\beta\sqrt{-1}$. Putting $m^2=-\lambda^2$ we have a quadratic to find λ^2. The roots of this quadratic must be real and positive.

If $A=B$, as in the case of a top spinning with its axis vertical, we have

$$\lambda=\pm\left\{\frac{2A-C}{2A}n\pm\sqrt{\frac{C^2n^2-4AMgh}{2A}}\right\}.$$

The motion is stable or unstable according as C^2n^2 is greater or less than $4AMgh$. If $C^2n^2=4AMgh$, the equation has equal roots and as the first minors are not zero the motion is unstable.

23. [As another example of the use of the modified Lagrangian function, let us consider a case discussed by Prof. Ball in the *Notices of the Royal Astronomical Society for March,* 1877. In a problem in Physical Astronomy, we want the *relative* co-ordinates of the system, while its *absolute* motion in space does not concern us. Lagrange's equations involve both the relative and absolute co-ordinates, and are therefore not particularly well adapted for such problems. By using the modified Lagrangian function, we may eliminate the absolute co-ordinates.

Let the system have n co-ordinates, let us choose as three of them the co-ordinates of the centre of gravity of the whole system, viz. θ, ϕ, ψ. There will then remain $n-3$ co-ordinates which are independent of these. Let T' be the kinetic energy of the system relative to its centre of gravity, V the potential energy, M the whole mass. Then the Lagrangian function is

$$L=\frac{1}{2}M(\theta'^2+\phi'^2+\psi'^2)+T'-V.$$

In problems in Physical Astronomy the potential energy is a function only of the *relative* positions of the bodies, and is therefore independent of θ, ϕ, ψ and their differential coefficients. We have therefore

$$\frac{dL}{d\theta'}=c_1,\quad \frac{dL}{d\phi'}=c_2,\quad \frac{dL}{d\psi'}=c_3.$$

Hence the modified Lagrangian function is

$$L'=T'-V-\text{a constant}.$$

It is independently clear that we might take this as the Lagrangian function, for the first three terms of L do not enter into any one of the Lagrangian equations, except the three formed by differentiating with regard to θ', ϕ', ψ'.

The function T' is made up of two parts, (1) the kinetic energies of the rotations of the bodies about their centres of gravity, which we may call T_1', and (2) the relative kinetic energies of the several bodies, each collected at its centre of gravity, which we may call T_2'. Let m_1, m_2, &c. be these masses; x_1, x_2, &c. the abscissæ of their centres of gravity referred to the centre of gravity of the whole as origin. Then, accents denoting differential coefficients with regard to the time, we have

$$m_1x_1' + m_2x_2' + \&c. = 0.$$

Let us square this and write

$$2x_1'x_2' = x_1'^2 + x_2'^2 - (x_1' - x_2')^2.$$

If we examine the coefficient of any power as $x_1'^2$ we see that it is

$$m_1^2 + m_1(m_2 + m_3 + \&c.) = m_1\Sigma m.$$

Hence the square becomes

$$\Sigma m\, \Sigma mx'^2 - \Sigma m_1m_2(x_1' - x_2')^2 = 0.$$

Similar expressions hold for the y and z co-ordinates. Hence on the whole we see that the relative kinetic energies of the several bodies collected at their respective centres of gravity is

$$T_2' = \frac{\Sigma m_1m_2v^2}{2\Sigma m},$$

where v is the relative velocity of the centres of gravity of the masses m_1, m_2. If we express this in any kind of co-ordinates, we may use the Lagrangian function L' to find the relative motion.

The expression for T_2' agrees with that given by Prof. Ball, but his demonstration is quite different. Prof. Cayley has given another demonstration in the same number of the Astronomical Notices.

The Lagrangian function thus found may be still further "modified." To avoid symbols of summation, let us consider the case of three particles moving in one plane under their mutual attractions. Let the separate masses be m_1, m_2, m_3, and let μ be their sum. Referring the system to m_1 as a central mass, let the distances of m_2, m_3 from m_1 be respectively r and ρ, and let the opposite side of the triangle be R. Let the interior angle between r and ρ be ϕ and the exterior angle between r and R be χ. Let θ be the angle r makes with some fixed straight line in space. We easily find

$$\mu T_2' = \frac{1}{2} A\theta'^2 + B\theta' + C,$$

where

$$A = m_1m_2r^2 + m_1m_3\rho^2 + m_2m_3R^2,$$
$$B = m_3(m_1\rho^2\phi' + m_2R^2\chi'),$$
$$C = \frac{1}{2\mu}\Sigma mm'v^2,$$

and v is the relative velocity of the masses m, m' calculated on the supposition that m_1 is fixed, and that the straight line r has no rotation round m_1. Thus A, B, C are all functions of r, ρ, ϕ and their differential coefficients with regard to the time.

If we only want the changes in the form and magnitude of the triangle joining the three particles, we may eliminate θ' by means of the equation

$$A\theta' + B = c_1.$$

We then find as our modified Lagrangian function

$$L'' = \frac{1}{\mu}\left\{C - \frac{1}{2}\frac{(C_1 - B)^2}{A}\right\} - V,$$

which contains only the *three* co-ordinates r, ρ and ϕ].

24. When the geometrical equations contain differential coefficients of the co-ordinates ξ, η, ζ, &c. of the system with regard to the time, we cannot express the co-ordinates x, y, z of any element of a body in terms of ξ, η, ζ, &c. by means of equations of the form

$$x = f_1(\xi, \eta, \zeta, \text{\&c.}, t),$$
$$y = f_2(\xi, \eta, \zeta, \text{\&c.}, t),$$
$$z = f_3(\xi, \eta, \zeta, \text{\&c.}, t).$$

It follows, as is pointed out in our books on Rigid Dynamics, that Lagrange's equations cannot be employed in the form

$$\frac{d}{dt}\frac{dT}{d\xi'} - \frac{dT}{d\xi} = -\frac{dV}{d\xi}.$$

In many of the most interesting problems in Rigid Dynamics, it so happens that the geometrical equations do contain $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, &c. For example, let a sphere be set rotating about a vertical diameter and be on the summit of a perfectly rough surface of any form. If a small disturbance be now given to it, the sphere may roll round and round the summit. During this motion the velocity of the point of contact is zero, and our mode of representing this analytically in terms of the co-ordinates will give us two equations of the form

$$A\xi' + B\eta' + C\zeta' + \ldots = 0.$$

To include such cases the equations of motion must be modified. If L be the difference between the kinetic and potential energies, all the Lagrangian equations may be written in the form

$$\left(\frac{d}{dt}\frac{dL}{d\xi'} - \frac{dL}{d\xi}\right)\delta\xi + \left(\frac{d}{dt}\frac{dL}{d\eta'} - \frac{dL}{d\eta}\right)\delta\eta + \text{\&c.} = 0,$$

where $\delta\xi$, $\delta\eta$, &c. are any small arbitrary displacements consistent with the geometrical equations. But if these geometrical equations be given in the form

$$\left.\begin{aligned} G = G_1\xi' + G_2\eta' + \ldots &= 0 \\ H = H_1\xi' + H_2\eta' + \ldots &= 0 \\ \&\text{c.} &= 0 \end{aligned}\right\},$$

these arbitrary displacements must satisfy

$$\left.\begin{aligned} G_1\,\delta\xi + G_2\delta\eta + \ldots &= 0 \\ \&\text{c.} &= 0 \end{aligned}\right\}.$$

If this were not the case, the geometrical displacement of the body given in applying Virtual Velocities would not be such as to cause the unknown frictional forces, &c. to disappear. Using the principle of Indeterminate Multipliers, we get

$$\left.\begin{aligned} \frac{d}{dt}\frac{dL}{d\xi'} - \frac{dL}{d\xi} + \lambda G_1 + \mu H_1 + \ldots &= 0 \\ \frac{d}{dt}\frac{dL}{d\eta'} - \frac{dL}{d\eta} + \lambda G_2 + \mu H_2 + \ldots &= 0 \\ \&\text{c.} &= 0 \end{aligned}\right\}.$$

These joined to the geometrical equations

$$\left.\begin{aligned} G_1\xi' + G_2\eta' + \ldots &= 0 \\ \&\text{c.} &= 0 \end{aligned}\right\},$$

are sufficient to determine the unknown co-ordinates ξ, η, &c. and the multipliers λ, μ, &c.

It will be more convenient to write these equations in the form

$$\frac{d}{dt}\frac{dL}{d\xi'} - \frac{dL}{d\xi} + \lambda\frac{dG}{d\xi'} + \mu\frac{dH}{d\xi'} + \ldots = 0,$$

$$\frac{d}{dt}\frac{dL}{d\eta'} - \frac{dL}{d\eta} + \lambda\frac{dG}{d\eta'} + \mu\frac{dH}{d\xi'} + \ldots = 0,$$

$$\&\text{c.} = 0,$$

the geometrical equations being

$$G = 0, \qquad H = 0, \qquad \&\text{c.}$$

It is of course obvious that these indeterminate coefficients λ, μ, &c. are merely the frictions or other resistances introduced into the equations in a convenient form.

25. In order to apply these equations to the oscillations of a system about a state of steady motion, it will be convenient to

change the co-ordinates ξ, η, &c. into others θ, ϕ, &c. which vanish in the steady motion. Let L be thus expanded in powers of θ, ϕ, &c. as explained in Art. 1, and let P and Q have the meaning given to them in Art. 5.

Let us then put

$$\xi' = \alpha + \theta', \quad \eta' = \beta + \phi', \quad \text{\&c.}$$
$$\lambda = \lambda_0 + \lambda_1, \quad \mu = \mu_0 + \mu_1, \quad \text{\&c.}$$

where α, β, λ_0, μ_0, &c. are the values of ξ', η', λ, μ, &c. in steady motion. The geometrical equations will then take the form

$$\left.\begin{aligned} G = G_1(\alpha + \theta') + G_2(\beta + \phi') + \text{\&c.} &= 0 \\ \text{\&c.} &= 0 \end{aligned}\right\},$$

and the equations connecting $\delta\theta$, $\delta\phi$, &c. will be

$$\left.\begin{aligned} G_1\delta\theta + G_2\delta\phi + \text{\&c.} &= 0 \\ \text{\&c.} &= 0 \end{aligned}\right\}.$$

In these equations G_1, G_2, &c. are functions of θ, ϕ, &c. Let a square bracket indicate that the value of the inscribed quantity in steady motion is to be taken. Thus $[G_1]$ means the value of G_1 when θ, ϕ, &c. have all been put zero.

The equations of steady motion may then, exactly as in Art. 3, be written

$$\left.\begin{aligned} -\left[\frac{dL}{d\theta}\right] + \lambda_0\left[\frac{dG}{d\theta'}\right] + \mu_0\left[\frac{dH}{d\theta'}\right] + \text{\&c.} &= 0 \\ -\left[\frac{dL}{d\phi}\right] + \lambda_0\left[\frac{dG}{d\phi'}\right] + \mu_0\left[\frac{dH}{d\phi'}\right] + \text{\&c.} &= 0 \\ \text{\&c.} &= 0 \end{aligned}\right\}.$$

From these the relations which exist between α, β, &c., λ_0, μ_0, &c. may be found.

The equations of the oscillatory motion may be written

$$\frac{d(P+Q)}{d\theta} - D\frac{dQ}{d\theta'} - \lambda_0 G_1 - \lambda_1\left[\frac{dG}{d\theta'}\right] - \mu_0 H_1 - \mu_1\left[\frac{dH}{d\theta'}\right] - \text{\&c.} = 0,$$

with similar equations for ϕ, ψ, &c.

26. The final determinant written for two variables θ, ϕ, and two geometrical equations G and H in the notation of Art. 19, will be

$$\begin{vmatrix} \begin{array}{c} B_{11}m^2 - A_{11} + E_{11} \\ + \lambda_0 \left[\dfrac{dG_1}{d\theta}\right] + \mu_0 \left[\dfrac{dH_1}{d\theta}\right] \end{array} & \begin{array}{c} B_{12}m^2 - A_{12} + E_{12} \\ + (C_{21} - C_{12})\, m \\ + \lambda_0 \left[\dfrac{dG_1}{d\phi}\right] + \mu_0 \left[\dfrac{dH_1}{d\phi}\right] \end{array} & \left[\dfrac{dG}{d\theta'}\right] & \left[\dfrac{dH}{d\theta'}\right] \\ \begin{array}{c} B_{12}m^2 - A_{12} + E_{12} \\ - (C_{21} - C_{12})\, m \\ + \lambda_0 \left[\dfrac{dG_2}{d\theta}\right] + \mu_0 \left[\dfrac{dH_2}{d\theta}\right] \end{array} & \begin{array}{c} B_{22}m^2 - A_{22} + E_{22} \\ + \lambda_0 \dfrac{dG_2}{d\phi} + \mu_0 \dfrac{dH_2}{d\phi} \end{array} & \left[\dfrac{dG}{d\phi'}\right] & \left[\dfrac{dH}{d\phi'}\right] \\ \left[\dfrac{dG}{d\theta}\right] + \left[\dfrac{dG}{d\theta'}\right] m & \left[\dfrac{dG}{d\phi}\right] + \left[\dfrac{dG}{d\phi'}\right] m & 0 & 0 \\ \left[\dfrac{dH}{d\theta}\right] + \left[\dfrac{dH}{d\theta'}\right] m & \left[\dfrac{dH}{d\theta}\right] + \left[\dfrac{dH}{d\phi'}\right] m & 0 & 0 \end{vmatrix} = 0.$$

It will be noticed how very much this determinant is simplified if the values of λ, μ in steady motion are zero.

27. Let us apply these equations to the solution of the following problem.

A heavy sphere rotating about a vertical diameter rests in equilibrium on the summit of a perfectly rough surface and being slightly disturbed makes small oscillations, find the periods.

As the sphere moves about, its centre always lies on a surface which may be called parallel to the given surface. Let the highest point of this surface be taken as the origin and let the axes of x and y be the tangents to its lines of curvature at O, so that the equation to the surface in the neighbourhood of O is

$$z = -\tfrac{1}{2}\left(\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}\right).$$

Let P be the centre of the sphere, PC that diameter which is vertical when the sphere is in equilibrium on the summit. Let PA, PB be two other diameters forming with PC a system of rectangular axes fixed in the sphere. Let the inclination of PC to the axis of Z, which is vertical, be θ, and let the vertical plane through PC make with the plane xz an angle ψ, and with the plane CPA an angle ϕ. The vis viva $2T$ of the sphere will then be given by

$$T = \tfrac{1}{2}(x'^2 + y'^2) + \tfrac{1}{2}k^2\{(\phi' + \psi' \cos\theta)^2 + \theta'^2 + \sin^2\theta\,\psi'^2\}.$$

Let $\sin\theta\cos\psi = \xi$, $\sin\theta\sin\psi = \eta$, then we have to the necessary degree of approximation

$$\left.\begin{aligned} \theta^2\psi' &= \xi\eta' - \eta\xi' \\ \theta'^2 + \sin^2\theta\psi'^2 &= \xi'^2 + \eta'^2 \end{aligned}\right\}.$$

Also let $\phi + \psi = \chi$. These transformations of co-ordinates are all permissible, because they do not involve any differential coefficients with regard to the time. We thus find if L be the difference between the kinetic and potential energies

$$L = \tfrac{1}{2}(x'^2 + y'^2) + \frac{k^2}{2}\{\chi'^2 - \chi'(\xi\eta' - \eta\xi') + \xi'^2 + \eta'^2\} + \frac{g}{2}\left(\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}\right).$$

If ω_x, ω_y, ω_z are the angular velocities of the sphere about parallels to the axes, the geometrical conditions are

$$\left.\begin{aligned} x' - a\left(\omega_y - \omega_z\frac{y}{\rho_2}\right) &= 0 \\ y' + a\left(\omega_x - \omega_z\frac{x}{\rho_1}\right) &= 0 \end{aligned}\right\},$$

where a is the radius of the sphere. These equations by well-known rules reduce to

$$\left.\begin{aligned} -\frac{x'}{a} + \phi'\sin\psi\sin\theta + \theta'\cos\psi - (\psi' + \phi'\cos\theta)\frac{y}{\rho_2} &= 0 \\ -\frac{y'}{a} - \phi'\cos\psi\sin\theta + \theta'\sin\psi + (\psi' + \phi'\cos\theta)\frac{x}{\rho_1} &= 0 \end{aligned}\right\};$$

expressing these in terms of our new co-ordinates we have

$$\left.\begin{aligned} G = -\frac{x'}{a} + \chi'\eta + \xi' - \chi'\frac{y}{\rho_2} &= 0 \\ H = -\frac{y'}{a} - \chi'\xi + \eta' + \chi'\frac{x}{\rho_1} &= 0 \end{aligned}\right\}.$$

The position of the system has now been expressed in terms of such co-ordinates, that the coefficients in the governing functions L, G, H are all constant. See Art. 2.

The steady motion is given by x, y, ξ, η all zero, and $\chi' = n$. To find λ_0, μ_0 we may use the equations of steady motion

$$-\left[\frac{dL}{dq}\right] + \lambda_0\left[\frac{dG}{dq'}\right] + \mu_0\left[\frac{dH}{dq'}\right] = 0,$$

where q stands for any one of the co-ordinates. Taking $q = x$ and $q = y$, we see that $\lambda_0 = 0$ and $\mu_0 = 0$.

To find the oscillation we may substitute in the determinant and thus form the equation which gives the periods. As most of the constituents of the determinants are zero, it will be more convenient to form each equation directly from the standard formula

$$\frac{d(P+Q)}{dq} - D\frac{dQ}{dq'} - \lambda_1\left[\frac{dG}{dq'}\right] - \mu_1\left[\frac{dH}{dq'}\right] = 0,$$

where q stands for any one of the co-ordinates. Taking q in turn to be x, y, χ, ξ, η we find

$$\left.\begin{aligned} x'' - g\frac{x}{\rho_1} - \frac{\lambda_1}{a} &= 0 \\ y'' - g\frac{y}{\rho_2} - \frac{\mu_1}{a} &= 0 \\ k^2\chi'' \qquad &= 0 \\ k^2(\xi'' + \chi'\eta') + \lambda_1 &= 0 \\ k^2(\eta'' - \chi'\eta') + \mu_1 &= 0 \end{aligned}\right\}.$$

Putting $\chi' = n$ these with the two geometrical equations are all linear and ready for elimination.

Eliminating λ_1, μ_1 we have

$$\left.\begin{aligned} \xi'' + n\eta' + \frac{a}{k^2}x'' - \frac{ga}{k^2}\frac{x}{\rho_1} &= 0 \\ \eta'' - n\xi' + \frac{a}{k^2}y'' - \frac{ga}{k^2}\frac{y}{\rho_2} &= 0 \end{aligned}\right\}.$$

Substituting for ξ, η from the two geometrical equations, we have

$$\left.\begin{aligned} \frac{k^2+a^2}{a^2}x'' - g\frac{x}{\rho_1} &= -\frac{nk^2}{a^2\rho_2}y' \\ \frac{k^2+a^2}{a^2}y'' - g\frac{y}{\rho_2} &= +\frac{nk^2}{a^2\rho_1}x' \end{aligned}\right\}.$$

To solve these put $x = X\cos(pt+q)$ $y = Y\sin(pt+q)$ so that p is the quantity required. We obviously have

$$\left(p^2 + \frac{a^2}{a^2+k^2}\frac{g}{\rho_1}\right)\left(p^2 + \frac{a^2}{a^2+k^2}\frac{g}{\rho_2}\right) = \frac{k^4}{(a^2+k^2)^2}\frac{a^2n^2}{\rho_1\rho_2}p^2,$$

which is a quadratic to find p^2.

If ρ_1, ρ_2 have opposite signs the roots cannot be real, and the steady motion must be unstable. If ρ_1, ρ_2 are both positive, so that the sphere is on the *summit*, the motion is stable only if

$$n^2 > \frac{a^2+k^2}{k^4}g(\sqrt{\rho_1}+\sqrt{\rho_2})^2.$$

CHAPTER V.

Certain subsidiary determinants are formed from the dynamical determinant, and it is shown that there must be at least as many roots indicating stability as there are variations of sign lost in these subsidiary determinants, and must exceed the number lost by an even number. *Arts.* 1—5, *and* 9.

This is equivalent to a maximum and minimum criterion of stability with similar limitations. *Arts.* 6—8.

Effect of equal roots on this test of the stability of the system. *Arts.* 10, 11.

Example. *Art.* 12.

1. In order to test the nature of the roots of the determinantal equation, let us apply a method analogous to that by which Dr Salmon in his Higher Algebra proves the reality of the roots of the equation which occurs in the determination of the secular inequalities of the planets.

2. Let Δ be the determinant which forms the left-hand side of the fundamental equation, let Δ_{pq} be the determinant formed by omitting the p^{th} row and q^{th} column. Let Δ_r be the determinant formed by omitting the first r rows and r columns. Thus $\Delta_1 = \Delta_{11}$. We then have by a known theorem in determinants

$$\Delta\Delta_2 = \Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}.$$

It has already been noticed that if we change m into $-m$, the determinant Δ is changed into another determinant whose columns and rows are the same as the rows and columns of the first determinant. It easily follows that the minor $-\Delta_{12}$ is changed into the minor $-\Delta_{21}$ by changing m into $-m$. Hence if

$$\Delta_{12} = \phi(m^2) + m\psi(m^2)$$

then

$$\left.\begin{aligned}\Delta_{12} &= \phi(m^2) + m\psi(m^2) \\ \Delta_{21} &= \phi(m^2) - m\psi(m^2)\end{aligned}\right\}.$$

Hence the product $\Delta_{12}\Delta_{21}$ is necessarily positive for all negative values of m^2.

It also follows that if Δ_{12} vanishes for any negative value of m^2 then Δ_{21} also vanishes for the same value of m^2.

3. When the determinant Δ_1 vanishes, we have

$$\Delta\Delta_2 = -\Delta_{12}\Delta_{21},$$

so that Δ and Δ_2 must have opposite signs, or one of them must be zero. Consider then the series of determinants

$$\Delta,\ \Delta_1,\ \Delta_2,\ \Delta_3,\ \ldots\ldots$$

each one being formed from the preceding by erasing the first row and the first column. We thus have a series of functions of m^2 whose degrees regularly diminish from the n^{th} to the first. As we may suppose the determinant Δ to have a row and a column of zeros added on at the bottom and right-hand side, but with any positive constant in the right-hand bottom corner, we may add to this series of determinants any positive constant. We have just proved that if any determinant of this series vanish for a negative value of m^2, the two determinants on each side have opposite signs. The case in which two successive determinants vanish for the same value of m^2 will be considered afterwards.

We may then use these determinants in a manner somewhat similar to that in which we use Sturm's functions, provided no two successive functions vanish for the same negative value of m^2. No variation of sign can be lost as we pass from $m^2 = -\infty$ to $m^2 = 0$ except by the vanishing of the determinant Δ at the head of the series. And when a variation of sign is lost, it will be regained again at the next root, unless a root of the determinant Δ_1 separates the two roots of the determinant Δ. *If therefore in this passage from* $-\infty$ *to zero, as many variations of sign are lost as is indicated by the highest power of* m^2, *the values of* m^2 *found from the determinantal equation must be all real and negative.* It will also follow that the roots of each of the series of determinants are all real and negative, and that the roots of each separate the roots of the determinant next above it. If, however, the proper number of variations of signs be not lost in the passage from $m^2 = -\infty$ to $m^2 = 0$, it does not follow that the values of m^2 are not real and negative.

4. If the proper number of variations of sign has not been lost in this passage from $m^2 = -\infty$ to $m^2 = 0$, this proposition does not leave us without information as to the nature of the roots. *We infer that the number of real negative values of* m^2 *is equal to or exceeds the number of variations of sign lost by an even number and unless the number of variations be even the system is unstable.*

If we do not object to the labour of expanding the determinants, we might extend this theorem to determine the positions of the negative values of m^2 as well as their number. The number of real negative values of m^2 between $m^2 = -\alpha$ and $m^2 = -\beta$ is equal to, or exceeds by an even number, the number of variations of sign lost in the series of determinants. In this form the theorem resembles Fourier's theorem in the Theory of Equations.

5. The converse of this proposition has not been proved. If in the passage from $m^2 = -\infty$ to $m^2 = 0$, the number of variations of signs is unaltered, it is not true that the values of m^2 cannot be real and negative. Thus in the simple case

$$\begin{vmatrix} m^2 - a_{11} & a_{12}m \\ -a_{12}m & m^2 - a_{22} \end{vmatrix} = 0,$$

where a_{11}, a_{12}, a_{22} are all positive, no variations are lost, yet if $a_{12} > \sqrt{a_{11}} + \sqrt{a_{22}}$ the values of m^2 are real and negative. And if $a_{12} = \sqrt{a_{11}} + \sqrt{a_{22}}$ the roots are equal and negative. It will be noticed that the minors in this last case are not zero.

6. In order to discover the meaning of these losses or gains of changes of sign, it will be convenient to make such changes of the co-ordinates as will simplify the dynamical determinant as much as possible. Let us write

$$V_0 = (E_{11} - A_{11}) \frac{\theta^2}{2} + (E_{12} - A_{12})\,\theta\phi + \ldots$$

If we now change our co-ordinates by writing for θ, ϕ, &c. linear expressions of some new co-ordinates, we know that we can clear this expression of all the terms containing the products. We know also that this can be done in an infinite number of ways. We thus have

$$V_0 = a_{11} \frac{\theta_1^2}{2} + a_{22} \frac{\phi_1^2}{2} + \ldots$$

where the symbols θ_1, ϕ_1, &c. represent the new co-ordinates.

Again, let us consider the expression

$$W_0 = B_{11} \frac{\theta'^2}{2} + B_{12}\,\theta'\phi' + \ldots\ldots$$

the coefficients are here the values of P, Q, &c. in the general expression for T, Art. 1, Chap. IV. when θ, ϕ, &c. are all put zero. But since T is necessarily positive for all values of ξ', $\eta'\ldots\xi$, $\eta,\ldots$ it follows that W_0 is positive for all values of θ', $\phi'\ldots$ Hence by a well-known theorem, we may by a *real* linear transformation of the variables clear the expression W_0 also of the terms containing the products θ', ϕ', &c. and can make the coefficients of the squares any positive constants we may please. We thus have

$$W_0 = \frac{\theta_1'^2}{2} + \frac{\phi_1'^2}{2} + \dots\dots$$

It may be shown that we cannot in general clear the expression for T of all the terms containing $\theta\phi'$, $\theta'\phi$, &c. unless $C_{12} - C_{21} = 0$, &c. by substituting for θ, ϕ, &c., any linear functions of other variables. As this is only a negative result of which no further use will here be made, it is unnecessary to supply the demonstration.

7. Using these simplifications the determinant Δ will now take the simpler form,

$$\Delta = \begin{vmatrix} m^2 + a_{11}, & a_{12}m, & a_{13}m, & \dots \\ -a_{12}m, & m^2 + a_{22}, & a_{23}m, & \dots \\ -a_{13}m, & -a_{23}m, & m^2 + a_{33}, & \dots \\ \dots\dots\dots, & \dots\dots\dots, & \dots\dots\dots, & \dots \end{vmatrix} = 0,$$

where $a_{12} = C_{21} - C_{12}$, &c.

If we form from Δ the series of subsidiary determinants

$$\Delta, \Delta_1, \Delta_2 \dots,$$

terminating with any positive constant, we see that when $m^2 = -\infty$, these subsidiary determinants are alternately positive and negative, and when $m^2 = 0$, they become

$$a_{11}a_{22}a_{33}\dots, \; a_{22}a_{33}\dots, \; a_{33}a_{44}\dots, \; \text{\&c.},$$

which are all positive if a_{11}, $a_{22}\dots$ are all positive. Hence if a_{11}, $a_{22}\dots$ are all positive, the proper number of changes of sign has been lost, and therefore the roots of the dynamical equation $\Delta = 0$ satisfy the condition of stability.

If a_{11}, a_{22}, &c. are all positive, we see that V_0 is a minimum for all variations of θ_1, ϕ_1, &c. and therefore for all variations of the original co-ordinates. If a_{11}, a_{22}, &c. are not all positive, there will be as many variations of sign lost as there are positive quantities in the series a_{11}, a_{22}, &c. In this case V_0 is a minimum for some variations of θ, ϕ, &c. and not for others. It is shown in the appendix to Williamson's *Differential Calculus*, that n independent conditions are necessary that a quadratic expression of n variables should be always positive. These are given in the form of determinants and may be briefly summed up, in the statement that

$$A_{11}x_1^2 + 2A_{12}x_1x_2 + \dots$$

is always positive if the roots of

$$\begin{vmatrix} A_{11}+\lambda, & A_{12}, & A_{13}, & \&c. \\ A_{12}, & A_{22}+\lambda, & A_{23}, & \&c. \\ A_{13}, & A_{23}, & A_{33}+\lambda, & \&c. \\ \&c. & \&c. & \&c. & \&c. \end{vmatrix} = 0,$$

are all real and negative.

8. We may now put the proposition of Art. 3 into another form. *Let* L *be the general expression for the excess of the kinetic energy over the potential energy of a dynamical system in terms of its* n *co-ordinates* x, y, &c. *Let this system be moving in steady motion with constant values of* $\frac{dx}{dt}, \frac{dy}{dt}$, &c. *Then if* L *be a maximum for all variations of* x, y, &c. *keeping* $\frac{dx}{dt}, \frac{dy}{dt}$, &c. *unchanged, then that steady motion is stable.*

If however only r *of the* n *conditions necessary to make* L *a maximum be satisfied, then the number of roots of the dynamical equation which satisfy the conditions of stability is equal to* r *or exceeds* r *by an even number. There cannot be stability unless* n − r *is an even number.*

If the system be oscillating about a position of equilibrium, $\frac{dx}{dt}, \frac{dy}{dt}$, &c. are all zero, and this leads at once to the condition, that the equilibrium is stable if the potential energy is a minimum.

9. In this reasoning, we have for convenience excepted the case in which two successive determinants in the series $\Delta, \Delta_1, \Delta_2 \ldots$ vanish for the same value of m^2. But this exception is of no real importance, for we may change these determinants into others whose constituents are very slightly different from those of the given determinants but which are such that no successive two of the series have a common root. In the limit, therefore, when these arbitrary changes of the constituents are indefinitely small, the roots of the series of determinants will still be real under the same circumstances as before, and the roots of each will separate, or *coincide with,* the roots of the next above it in the series.

To show that these changes are possible, let us consider the row of determinants beginning at the last. The determinant Δ_n is a positive constant, the next Δ_{n-1} is $m^2 + a_{nn}$. Proceeding thus, suppose we arrive at two determinants which we may call Δ_1 and Δ which have a common root. If we now change the constituents a_{11}, a_{12}, a_{13}, &c. into $a_{11} + \delta a_{11}$, $a_{12} + \delta a_{12}$, &c. we do not alter Δ_1, but, except for the root $m^2 = 0$, we *do* alter Δ in an arbitrary manner.

When for example a_{13} is altered, we alter a constituent both in the first row and in the first column. Since

$$\Delta\Delta' = \Delta_1\Delta_{33} - \Delta_{13}\Delta_{31},$$

where Δ' is the determinant formed from Δ by omitting the first and third rows and columns, we see that when Δ and Δ_1 both vanish, the product $\Delta_{13}\Delta_{31}$ and therefore by Art. 2 both Δ_{13} and Δ_{31} must vanish. The determinant Δ is therefore altered by $-\Delta_2(\delta a_{13})^2$ which does not vanish, since Δ_2 is by hypothesis finite for the particular value of m^2 under consideration.

If any determinant of the series vanishes when $m^2 = 0$, it is clear that one of the quantities a_{11}, a_{22}... must be zero. If we replace this by any small positive quantity, the argument will apply as before.

10. It is important to consider the effect of *equal roots* on the test of stability given in Art. 8 of this chapter.

In this case we know that the roots of the minor Δ_{11} separate the roots of Δ. If therefore Δ have two negative equal roots, it is clear that Δ_{11} must have one of them. In the same way Δ_{22}, Δ_{33}, &c. must also all vanish for this value of m. Since

$$\Delta\Delta' = \Delta_{pp}\Delta_{qq} - \Delta_{pq}\Delta_{qp},$$

it follows as in Art. 2 if Δ and Δ_{pp} both vanish, that Δ_{pq} and Δ_{qp} also vanish. Hence all the first minors of the determinant Δ vanish. This is the case considered in Art. 5 of Chap. I. The equal roots instead of introducing into the expressions for the coordinates terms which contain t as a factor merely render two of the coefficients, instead of one, indeterminate.

In the same way if the equation $\Delta = 0$ is satisfied by three values of m^2 equal to the same negative quantity, the equation $\Delta_{pp} = 0$ must have two of them, and its principal minor must have one of them. Reasoning as before we see that all the second minors of Δ must be zero. This is again the test that there should be no terms which contain t as a factor.

The presence therefore of equal roots does not in the theorem of Art. 8 affect the stability of the motion.

When a system is disturbed from a position of equilibrium whether stable or unstable, the roots of the fundamental determinant are separated by its minors in the manner described in Art. 8 of this Chapter. By what has just been proved, we see that if the fundamental determinant have equal roots, whether positive or negative, these do not introduce into the integrals terms which contain t as a factor.

11. The proposition in the last article may be made more general. If the fundamental determinant be reduced to the form indicated in Art. 7, we shall show that if Δ vanish for two equal negative values of m^2 which are numerically greater than the greatest negative quantity in the series a_{11}, a_{22}, a_{33}, &c., then these equal roots will not introduce any terms into the solution with t as a factor. If a_{11}, a_{22}, &c. are all positive, this reduces to the proposition proved in the last article.

Following the same notation as before, we have

$$\Delta\Delta' = \Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}.$$

If neither Δ_{11} nor Δ_{22} are zero, they must have the same sign when Δ vanishes for a negative value of m^2. For their product is equal to $\Delta_{12}\Delta_{21}$ which has been proved in Art. 2 to be positive. Hence all the leading first minors, viz. Δ_{11}, Δ_{22}, &c. must have the same sign for any negative value of m^2 which makes Δ vanish.

By differentiation we have

$$\frac{d\Delta}{dm} = 2m\Delta_{11} + a_{12}\Delta_{12} + \ldots - a_{12}\Delta_{21} + 2m\Delta_{22} + \ldots - \&c.$$

But we have also

$$\begin{aligned} \Delta &= (m^2 + a_{11})\,\Delta_{11} + a_{12}m\,\Delta_{12} + \ldots \\ \Delta &= -\,a_{12}m\,\Delta_{21} + (m^2 + a_{22})\,\Delta_{22} + \ldots \\ \&c. &= \&c. \end{aligned}$$

Hence if n be the highest power of m occurring in Δ, we have

$$m\frac{d\Delta}{dm} = n\Delta + (m^2 - a_{11})\,\Delta_{11} + (m^2 - a_{22})\,\Delta_{22} - \&c.$$

If then Δ and $\frac{d\Delta}{dm}$ both vanish for any negative value of m^2 greater than the greatest negative quantity in the series a_{11}, a_{22}, &c., we have the sum of a number of quantities all of the same sign equal to zero. This requires that each should be zero. We have therefore $\Delta_{11} = 0$, $\Delta_{22} = 0$, &c. The rest of the proof is the same as before.

By differentiating the expression for $\frac{d\Delta}{dm}$ and substituting for $\frac{d\Delta_{11}}{dm}$, $\frac{d\Delta_{22}}{dm}$, &c. their values in terms of their first leading minors, we may extend this proposition to the case in which the fundamental determinant has three equal roots, and so on.

12. *A sphere is suspended by a string* OA *from a fixed point* O, *and is set rotating about a vertical diameter which is in the same straight line as the string with an angular velocity* n. *A small disturbance is given, determine if the steady motion is stable.*

Let O be the origin, and let the axis of z be vertically downwards, let lx, ly, l be the co-ordinates of A, the point at which the string is attached. Let C be the centre, and let $a\xi$, $a\eta$, a be the co-ordinates of C relative to A. Then, exactly as in Chap. IV. Art. 22, the Lagrangian function may be shown to be

$$L = \frac{k^2}{2}\left\{\left(\chi' - \frac{\xi\eta'}{2} + \frac{\xi'\eta}{2}\right)^2 + \xi'^2 + \eta'^2\right\}$$

$$+ \tfrac{1}{2}(a\xi' + lx')^2 + \tfrac{1}{2}(a\eta' + ly')^2 - g\left\{l\frac{x^2+y^2}{2} + a\frac{\xi^2+\eta^2}{2}\right\},$$

the mass being taken unity.

Putting $\chi' = n$, ξ', η', x', y' all zero, we see that L is a maximum when x, y, ξ, η are zero, the steady motion is therefore stable for all values of n.

If we put $k^2 = \frac{2}{5}a^2$, and $m^2 = -\lambda^2$, so that x, y, &c. are all represented by terms of the form $\Sigma A \cos(\lambda t + \alpha)$, we may, by the methods of the last chapter, prove

$$\left(\lambda^2 - \frac{g}{l}\right)\left(\lambda^2 \pm n\lambda - \frac{5l}{2a}\right) = \frac{5g}{2l}\lambda^2.$$

This equation, whatever may be the sign of n, has two positive and two negative roots. All four give stable oscillations.

CHAPTER VI.

If the energy of the system be a maximum or minimum under certain conditions, the motion whether steady or not is stable. Arts. 1—3.

When the motion is steady, it will be also stable if a certain function of the co-ordinates called V + W *is a minimum. Art.* 4.

If there be only one co-ordinate which enters into the Lagrangian function, except as a differential coefficient, this condition is necessary and sufficient. Arts. 5, 6.

Additional conditions when there are two co-ordinates. Art. 8.

1. When a system is oscillating about a position of equilibrium, it is well known that we may determine the stability or instability of the equilibrium by what we may call the "energy criterion." This criterion may also be sometimes used when the system is oscillating about a state of steady motion.

Let E be the sum of the kinetic and potential energies of the system. Then throughout any motion of the system we have

$$E = h,$$

where h is a constant depending on the initial conditions. If θ, ϕ, &c. are the co-ordinates of the system, E is a known function of $\theta, \theta', \phi, \phi'$, &c. Suppose that some of the other first integrals of the equations of motion are known. Let these be

$$\left.\begin{array}{c} F_1(\theta, \theta', \phi, \phi', \&c.) = C_1 \\ F_2(\theta, \theta', \phi, \phi', \&c.) = C_2 \\ \&c. \end{array}\right\},$$

the time t being absent.

For the purposes of this proposition let us suppose $\theta, \theta', \phi, \phi'$, &c. to be separate variables unconnected with each other except by the equations just written down.

If E be an absolute maximum or an absolute minimum for all variations of θ, θ', &c., those corresponding to the given motion making E constant, then that motion is stable for all displacements which do not alter the constants C_1, C_2, &c.

If this proposition be not evident, it may be proved by eliminating as many of the letters as possible. If θ, θ', &c. be the remaining co-ordinates we have

$$E = f(\theta, \theta', \text{\&c. } C_1, C_2, \ldots).$$

Let h be the value of E in the given motion, and let the system be started in some slightly different manner so that

$$E = h + \delta h.$$

If E be a maximum along the given motion, then any change whatever in θ, θ', &c. decreases E. Hence θ, θ', &c. cannot deviate so much from their values along the given motion that the change in E becomes greater than δh.

2. Let us apply this principle to a system of bodies which moves in steady motion with some co-ordinates θ, ϕ, &c. such that their differential coefficients θ', ϕ', &c. are constant, and the remaining co-ordinates ξ, η, &c. themselves constant. Let us further suppose that the energy is a function of θ', ϕ', &c., but not of θ, ϕ. By Lagrange's Equations we have the integrals

$$\frac{dT}{d\theta'} = C_1, \quad \frac{dT}{d\phi'} = C_2, \text{ \&c.}$$

It is clear that the system can describe any one of a number of steady motions, which we have already called parallel motions, and which are determined by

$$\theta' = p, \qquad \phi' = q, \text{ \&c.}$$
$$\xi = \alpha, \qquad \eta = \beta, \text{ \&c.}$$

where p, q, &c. α, β, &c. are constants which satisfy all Lagrange's equations.

We have thus as many relations between these constants as there are co-ordinates ξ, η, &c. Let the system be started with any initial conditions we please, then the constants C_1, C_2, ... are given. These being known we have as many relations between the constants of steady motion as there are co-ordinates θ, ϕ, &c. The steady motion is therefore determined. If the energy of this initial motion is nearly equal to that of this steady motion, and if it be a maximum or minimum as explained above, then the system will never deviate far from its corresponding position in the steady motion, and this steady motion may be called stable.

3. Example. *A top is set spinning on its point on a perfectly rough horizontal ground, with its axis inclined to the vertical, find the condition of stability.*

Let θ be the inclination of the axis OC of the top to the vertical OZ; ψ the inclination of the plane ZOC to a vertical

6—2

plane fixed in space, and ϕ the inclination to a plane through OC fixed in the body. Let O be the apex, G the centre of gravity which lies in OC, $h = OG$. Let A, A, C be the principal moments of inertia at O, and M the mass of the top. We have then

$$E = \frac{C}{2}(\phi' + \psi' \cos\theta)^2 + \frac{A}{2}(\theta'^2 + \sin^2\theta\psi'^2) + Mgh\cos\theta.$$

By Lagrange's equations we have the integrals

$$\phi' + \psi'\cos\theta = n,$$
$$Cn\cos\theta + A\sin^2\theta\psi' = m,$$

where n and m are two constants, the former representing the angular velocity of the top about its axis, and the latter the angular momentum about the vertical. If we now eliminate ϕ' and ψ' we find that E is a minimum when $\theta = \alpha$, if

$$C^2n^2 > 4MghA\cos\alpha,$$

which is the result given by other methods.

4. The theorem of Art. 2 may be put into another form. Let the kinetic energy be

$$T = T_{\theta\theta}\frac{\theta'^2}{2} + T_{\theta\phi}\theta'\phi' + \dots$$

Then since θ, ϕ, &c. are absent from the coefficients, we have the integrals

$$T_{\theta\theta}\theta' + T_{\theta\phi}\phi' + \dots = C_1 - T_{\theta\xi}\xi' - T_{\theta\eta}\eta' - \&c.$$
$$T_{\theta\phi}\theta' + T_{\phi\phi}\phi' + \dots = C_2 - T_{\phi\xi}\xi' - T_{\phi\eta}\eta' - \&c.$$
$$\&c. = \&c.$$

For the sake of brevity let us call the right-hand sides of these equations $C_1 - X$, $C_2 - Y$, &c. Since T is a homogeneous function of θ', ϕ', &c., we have, as in Chap. IV. Art. 21,

$$T = T_{\xi\xi}\frac{\xi'^2}{2} + T_{\xi\eta}\xi'\eta' + \dots$$
$$+ \tfrac{1}{2}\theta'(C_1 + X) + \tfrac{1}{2}\phi'(C_2 + Y) + \dots$$

If we substitute in the second line the values of θ', ϕ', &c. found by solving the linear equations just written down, we have the determinant

$$-\begin{vmatrix} 0, & C_1 + X, & C_2 + Y, & \&c. \\ C_1 - X, & T_{\theta\theta}, & T_{\theta\phi}, & \&c. \\ C_2 - Y, & T_{\theta\phi}, & T_{\phi\phi}, & \&c. \\ \&c. & \&c. & \&c. & \&c. \end{vmatrix}\frac{1}{2\Delta}$$

where Δ is the discriminant of the terms in T which contain θ', ϕ', &c. This determinant is unaltered by changing the signs of X, Y, &c. and is a quadratic function of C_1, C_2, &c., X, Y, &c. Hence the terms C_1X, C_2Y, &c. do not occur. If then we put

$$W = -\begin{vmatrix} 0 & C_1, & C_2, & \&c. \\ C_1, & T_{\theta\theta}, & T_{\theta\phi}, & \&c. \\ C_2, & T_{\theta\phi}, & T_{\phi\phi}, & \&c. \\ \&c. & \&c. & \&c. & \&c. \end{vmatrix} \frac{1}{2\Delta}$$

we have

$$T = B_{\xi\xi}\frac{\xi'^2}{2} + B_{\xi\eta}\xi'\eta' + \&c. + W,$$

where $B_{\xi\xi}$, &c. are independent of C_1, C_2, &c. Now T is essentially positive for all values of the variables, and therefore for such as make C_1, C_2, &c. all zero. Hence the terms involving ξ', η', &c. are together a minimum when ξ', η', &c. are all zero. The coefficients $B_{\xi\xi}$, &c. may all be treated as constants since ξ', η', &c. are all small quantities.

If V be the potential energy, we have therefore the following rule. *If* W + V *be a minimum for all variations of* ξ, η, &c. *then the steady motion is certainly stable.* It should be noticed that $W + V$ is a function only of ξ, η, &c. the co-ordinates which are constant in the steady motion.

5. *If the energy be a function of one only of the co-ordinates, though the differential coefficients of all the others enter into its value, this condition is sufficient and necessary.*

Let ξ be this co-ordinate. Then by vis viva we have

$$B_{\xi\xi}\frac{\xi'^2}{2} + W + V = h.$$

Differentiating we have

$$B_{\xi\xi}\xi'' + \frac{d(W+V)}{d\xi} = 0.$$

This equation must be satisfied by the steady motion represented by $\xi = \alpha$. The second term $\dfrac{d(W+V)}{d\xi}$ must therefore vanish when $\xi = \alpha$, so that $W + V$ is a maximum or minimum. To find the oscillation let us put $\xi = \alpha + x$, we find

$$B_{\xi\xi}\frac{x'^2}{2} + \left[\frac{d^2(W+V)}{d\xi^2}\right]x = 0,$$

where the square bracket implies that ξ is to be put equal to α after differentiation. By the same reasoning as before $B_{\xi\xi}$ is necessarily positive, *and the motion will be stable or unstable according as* (W + V) *is a minimum or maximum.*

6. If we refer to Art. 21 of Chap. IV. we see that this function $W+V$ is the value of the modified Lagrangian function L' when ξ', η', &c. are all put zero, and the sign of the whole function changed. It therefore follows by Chap. V. that when $W+V$ is a minimum the steady motion is stable. The "energy criterion of stability," as far as it applies to steady motion, may therefore be deduced from that given in Chap. V. Art. 8. But the mode of demonstration adopted in that chapter gives us more information as the nature of the motion, while the modes of application to examples of the two criteria are quite different. The energy criterion may also be sometimes applied to determine the stability of a motion which is not steady.

[The relation between the theorem in this Chapter in which $E=T+V$ is made a minimum to that given in Chapters IV. and V. in which $L=T-V$ is made a maximum may be more distinctly perceived by the following statement.

Let x, y, &c. be the co-ordinates of the system, and let L be the Lagrangian function* so that $L=T-V$, then by Art. 3 of Chap. IV. the co-ordinates in steady motion satisfy the equations

$$\frac{dL}{dx}=0, \quad \frac{dL}{dy}=0, \text{ \&c.} \qquad\qquad (1).$$

Here L is expressed as a function of x, y, &c. x', y', &c.

Suppose some of the co-ordinates as θ, ϕ, &c. are absent from the expression for L, so that L is a function of θ', ϕ', &c., the remaining co-ordinates, viz. ξ, η, &c. and their differential coefficients. Then if we form the modified Lagrangian function as in Art. 21 of Chap. IV. the equations (1) of steady motion become

$$\frac{dL'}{d\xi}=0, \quad \frac{dL'}{d\eta}=0, \text{ \&c.} \qquad\qquad (2).$$

Here, as in the Hamiltonian equations C_1, C_2, &c. are the θ, ϕ, &c. components of momentum, and L' is expressed as a function of C_1, C_2, &c. ξ, η, &c. ξ', η', &c.

* Let u, v, &c. be the x, y, &c. components of momentum, so that $u=\frac{dT}{dx'}$, &c. and let H be the Hamiltonian function. Then $H=T+V$ and we easily deduce from the Hamiltonian equations that, in steady motion,

$$\frac{dH}{dx}=0, \quad \frac{dH}{dy}=0, \text{ \&c.}$$

Here H is expressed as a function of x, y, &c. u, v, &c.

But
$$L' = T - C_1\theta' - C_2\phi' - \&c. - V$$
$$= - T + \frac{dT}{d\xi'}\xi' + \frac{dT}{d\eta'}\eta' + \&c. - V,$$

by Euler's theorem on homogeneous functions. In steady motion ξ', η', &c. all vanish, hence the equations (2) become

$$-\frac{d(W+V)}{d\xi} = 0, \quad -\frac{d(W+V)}{d\eta} = 0, \ \&c. \ldots\ldots\ldots\ldots (3),$$

where W is the value of T when ξ', η', &c. are all put equal to zero. Here W is expressed as a function of C_1, C_2, &c. ξ, η, &c.

It is shown in Chap. V. that if the Lagrangian function expressed as required in equations (1) be a maximum the motion is stable. It is shown in this Chapter that if the function $W + V$ be a minimum the motion is stable.]

7. *To find the condition of stability when the Lagrangian function is a function of two only of the co-ordinates, though the differential coefficients of all the others enter into its value.*

Let ξ, η be these two co-ordinates, then the modified Lagrangian function as explained in Art. 20 of Chap. IV. will be a function of ξ, η, ξ', η' only.

Let the steady motion be given by $\xi = \alpha$, $\eta = \beta$, with the corresponding values of the other co-ordinates θ, ϕ, &c. Then α and β are constants. Let $\xi = \alpha + x$, $\eta = \beta + y$, and let us expand the modified Lagrangian function in powers of x, y. Neglecting the terms of the first order, as they only give the steady motion, let

$$L' = B_{11}\frac{x'^2}{2} + B_{12}x'y' + B_{22}\frac{y'^2}{2}$$
$$+ A_{11}\frac{x^2}{2} + A_{12}xy + A_{22}\frac{y^2}{2}$$
$$+ C_{11}xx' + C_{12}xy' + C_{21}yx' + C_{22}yy'.$$

Also let $E = C_{21} - C_{12}$. Then the condition of stability is that the roots of the following equation should be of the form $\beta\sqrt{-1}$.

$$\begin{vmatrix} B_{11}m^2 - A_{11}, & B_{12}m^2 - A_{12} + Em \\ B_{12}m^2 - A_{12} - Em, & B_{22}m^2 - A_{22} \end{vmatrix} = 0.$$

If
$$\left.\begin{aligned} \Delta' &= B_{11}B_{22} - B_{12}^2 \\ \Delta &= A_{11}A_{22} - A_{12}^2 \end{aligned}\right\},$$

and
$$\Theta = A_{11}B_{22} + A_{22}B_{11} - 2A_{12}B_{12}$$

be the two discriminants and the other invariant, this leads at once to the conclusion that the motion is stable only when

(1) Δ is positive,

(2) $E^2 - \Theta$ is positive and $> 2\sqrt{\Delta\Delta'}$.

If $A_{11}\frac{x^2}{2} + A_{12}xy + A_{22}\frac{y^2}{2}$ is a maximum when x and y are zero, the two conditions are obviously satisfied.

This condition may be otherwise expressed; if L' be the modified Lagrangian function, a steady motion is given by

$$\frac{dL'}{d\xi} = 0, \quad \frac{dL'}{d\eta} = 0, \quad \xi' = 0, \quad \eta' = 0.$$

This motion will be stable if for the values of ξ, η thus found,

(1) $\frac{d^2L'}{d\xi^2}\frac{d^2L'}{d\eta^2} - \frac{d^2L'}{d\xi d\eta}$ is positive,

(2) $\left(\frac{d^2L'}{d\xi d\eta'} - \frac{d^2L'}{d\xi' d\eta}\right)^2 - \left\{\frac{d^2L'}{d\xi^2}\frac{d^2L'}{d\eta'^2} + \frac{d^2L'}{d\eta^2}\frac{d^2L'}{d\xi'^2} - 2\frac{d^2L'}{d\xi d\eta}\frac{d^2L'}{d\xi' d\eta'}\right\}$

is positive and greater than

$$2\left\{\frac{d^2L'}{d\xi^2}\frac{d^2L'}{d\eta^2} - \frac{d^2L'}{d\xi d\eta}\right\}^{\frac{1}{2}} \cdot \left\{\frac{d^2L'}{d\xi'^2}\frac{d^2L'}{d\eta'^2} - \frac{d^2L'}{d\xi' d\eta'}\right\}^{\frac{1}{2}}.$$

8. The nature of the motion when thus reduced to depend on two co-ordinates may be illustrated by geometrical reasoning. Let the position be defined by two co-ordinates x, y which are zero along the steady motion. Let these be regarded as the co-ordinates of a point P referred to any axes. Then the motion of P exactly represents that of the system.

Let us construct the conics

$$A_{11}\frac{x^2}{2} + A_{12}xy + A_{22}\frac{y^2}{2} = a,$$

$$B_{11}\frac{x^2}{2} + B_{12}xy + B_{22}\frac{y^2}{2} = b.$$

Then, exactly as in Arts. 11—14 of Chap. IV., it will be found convenient to transform the co-ordinates by writing

$$\left.\begin{aligned} x &= a_1p + b_1q \\ y &= a_2p + b_2q \end{aligned}\right\}.$$

If μ be the modulus of transformation we have $\mu = a_1b_2 - a_2b_1$. It is easy to see by actual substitution, if E' be the difference of the coefficients of pq' and $p'q$, that

$$E' = \mu E.$$

If the transformation be from one set of oblique co-ordinates to another, let ω, ω' be the angles between the axes. We then have

$$\frac{E'}{\sin \omega'} = \frac{E}{\sin \omega}.$$

Transforming the axes to the common conjugate diameters, the conics become

$$\left.\begin{aligned} A_{11}'\frac{p^2}{2} + A_{22}'\frac{q^2}{2} = a \\ B_{11}'\frac{p^2}{2} + B_{22}'\frac{q^2}{2} = b \end{aligned}\right\},$$

the signs of a and b being such as to make these conics real. The equation to find m becomes

$$(B_{11}'m^2 - A_{11}')(B_{22}'m^2 - A_{22}') + E'^2m^2 = 0.$$

It is therefore necessary for stability that the conic a should be an ellipse as well as the conic b. It is also necessary that

$$\frac{E'}{\sqrt{B_{11}'B_{22}'}} > \sqrt{\frac{A_{11}'}{B_{11}'}} + \sqrt{\frac{A_{22}'}{B_{22}'}},$$

both roots having the same sign and the inequality being numerical.

Let OP, OP'; OQ, OQ' be the common conjugates of the two conics, this condition then becomes

$$\frac{E}{\sin \omega} \cdot \frac{\text{area of conic } b}{\pi \sqrt{ab}} > \frac{OQ}{OP} + \frac{OQ'}{OP'}.$$

When the system describes an oscillation with *one* period, *i.e.* an harmonic oscillation, the path of the representative particle is easily seen to be

$$(B_{11}''m^2 - A_{11}'')p^2 + (B_{22}''m^2 - A_{22}'')q^2 = \text{constant}.$$

The harmonic paths are therefore ellipses. It also appears that the two ellipses which represent the two harmonic vibrations and the two ellipses a and b have, all four, a common set of conjugate diameters.

CHAPTER VII.

Any small term of a high order, if its period is nearly the same as that of an oscillation of the system, may produce important effects on the magnitude of the oscillation. Art. 1.

Origin of such terms, with an example. Arts. 2—3.

Supposing the roots of the determinantal equation to satisfy the conditions of stability to a first approximation, yet if a commensurable relation hold between these roots it is necessary to examine certain terms of the higher orders to determine whether they will ultimately destroy the stability of the system. Art. 4.

If a certain relation hold among the coefficients of these terms, they will not affect the stability of the system, but only slightly alter the periods of oscillation. Arts. 5—7.

Examples, the first taken from Lagrange's method of finding the oscillations about a position of equilibrium. Arts. 8—9.

If the coefficients of the equations of motion should not be strictly constant, but only nearly so, the stability will not be affected, unless the reciprocals of their periods have commensurable relations with the reciprocals of the periods of oscillation of the system. Art. 10.

1. If we understand that a motion is called stable when any small disturbance does not cause the system to deviate far from its undisturbed motion, it is clear that we cannot be certain of the stability without examining the terms of the second order. It is possible that some of these may have their periods so timed that their effects may accumulate until the motion is changed.

Returning to the equations of Art. 3, Chap. I. we shall have on the right hand, instead of zero, a series of small terms of orders higher than the first. To find a second approximation, we substitute the values of x, y, &c. given at the end of Art. 3, in these terms.

They will therefore take the form Ne^{nt} and will produce in x, y, &c. terms of the form $\frac{N'}{f(n)}e^{nt}$, where N' is of the same order at least as the term considered, and $f(m)$ has the same meaning as in Chap. I. These will have to be expressed in trigonometrical real forms, but it is unnecessary to exhibit the process, for we see at once that no small term or force (whatever it may be called) of a high order can affect the stability of the motion unless it makes $f(n)$ very nearly or exactly equal to zero. In this case its period is very nearly or exactly equal to one of the periods of the motion given by taking terms of the first order only.

A remarkable use of this principle was made by Captain Kater in his experiments on the magnitude of gravity. It was important to determine if the support of his pendulum was perfectly firm. He tells us that he had recourse to a delicate and simple instrument the sensibility of which was so great that had the slightest motion taken place in the support it must have been instantly detected. The instrument consists of a steel wire the lower part of which, inserted in the piece of brass which serves as its support, is flattened so as to form a delicate spring. On the wire a small weight slides by means of which it may be made to vibrate in the *same time* as the pendulum to which it is to be applied as a test. When thus adjusted it is placed in the material to which the pendulum is attached, and should this not be perfectly firm its motion will be communicated to the wire, which in a little time will accompany the pendulum in its vibrations. This ingenious contrivance appeared fully adequate to the purpose for which it was employed, and afforded a satisfactory proof of the stability of the point of suspension. See *Phil. Trans.*, 1818.

2. Since the term Ne^{nt} is obtained by compounding the different terms in the values of x, y, &c. it is clear that

$$n = pm_1 + qm_2 \ldots$$

where p, q, &c. are positive integers whose sum is the order of the term. It is therefore only when the roots of the dynamical equation $f(m) = 0$ are such that a linear relation of the form

$$pm_1 + qm_2 + \ldots = m_1 \text{ very nearly}$$

exists between them, that we may expect to find important terms among the higher orders. The order of the terms to be examined will be $p + q + \ldots$, and unless this be also small, the terms will probably remain insignificant. If the root m_1 should occur twice in $f(m) = 0$ it is clear that the divisor $f(n)$ will be a small quantity of the second order, and the term may be said (as in the Lunar Theory) to rise two orders.

3. To take an example, let us suppose a particle to be describing an ellipse about a fixed centre of force in one focus. If disturbed it will describe a slightly different ellipse. Since

$$r = a\{1 - e\cos(nt + \epsilon - \omega) + \ldots\},$$
$$\theta = nt + \epsilon + 2e\sin(nt + \epsilon - \omega) + \ldots$$

we see that a slight change in the elements will cause variations in r and θ of the period $\frac{2\pi}{n}$, an additional variation in θ of the form $t\delta n + \delta\epsilon$ and an additional variation in r of the form δa. All these variations should by Art. 3 of Chap. I. be indicated by expressions of the form

$$r = \Sigma M e^{mt}, \quad \theta = \Sigma M' e^{mt},$$

where the values of m are the roots of the equation $f(m) = 0$. The roots therefore of the equation $f(m) = 0$ for δr are $m = 0$ and $\pm n\sqrt{-1}$, and for $\delta\theta$ are $m = 0$, 0 and $\pm n\sqrt{-1}$. We therefore infer that any small disturbing causes of the second order whose periods are nearly equal to that of the particle, will cause important inequalities in both δr and $\delta\theta$, and (since $f(m) = 0$ has two roots equal to zero) any term of long period will rise two orders in $\delta\theta$.

4. If the roots of the subsidiary equation are such that the relation

$$pm_1 + qm_2 + \ldots = m_1$$

holds accurately, the solution changes its character. We have now in the value of x a term of the form $te^{m_1 t}$. Unless the real part of m_1 is negative, this indicates that the system will depart widely from the motion which we took as a first approximation. We must therefore modify our first approximation (as in the Lunar Theory) by including in it the terms which produced these important effects. We may then enquire how far this modified first approximation indicates that the motion is stable or unstable. When these terms are included the equations to be solved are in general no longer linear, and it is sometimes impossible to find a solution sufficiently accurate to serve as a first approximation throughout the whole motion.

5. In some cases, however, the oscillations may still be represented by expressions of the form

$$x = M_1 e^{n_1 t} + M_2 e^{n_2 t} + \ldots$$
$$y = M_1' e^{n_1 t} + M_2' e^{n_2 t} + \ldots$$
&c.,

where the values of n_1, n_2... differ but slightly from the roots of the equation $f(m) = 0$. Let us investigate the condition that this should be true, and also determine whether the changes in the values of m_1, m_2... are sufficient to affect the stability.

Suppose that we have completed our first approximation, and find on proceeding to a higher approximation that the terms

$$N_1 e^{m_1 t} + N_2 e^{m_2 t} + \ldots$$
$$N_1' e^{m_1 t} + N_2' e^{m_2 t} + \ldots$$

present themselves on the right-hand side of the first set of equations in Art. 3, Chap. I. These terms are supposed to have arisen from several relations of the form

$$pm_1 + qm_2 + \ldots = m_1.$$

If these terms can be included in the first approximation by writing n_1, n_2, &c. for m_1, m_2, &c. we have, by substitution in the differential equations, certain equations connecting n, M, M', &c., whose left-hand sides are the same as those used in Art. 3 with n_1 written for m_1, but on the right-hand sides we have instead of zero the quantities N_1, N_1', &c. The test of the success of the process is that these modifications in the values of m must satisfy the same relations as before.

Now N_1, N_1', &c. are all at least of the second order of small quantities, hence up to terms of the first order the ratios M_1, M_1', &c. will be the same as before, so that we may put

$$M_1 = L_1 a_1, \;\; M_1' = L_1 b_1, \;\; \text{\&c.},$$

following the same notation as in Art. 3, Chap. I. We also have

$$M_1 f(n_1) = N_1 a_1 + N_1' a_1' + \ldots$$

Let $n_1 = m_1 + \delta m_1$ we find

$$\delta m_1 = \frac{N_1 a_1 + N_1' a_1' + \ldots}{f'(m_1) L_1 a_1}.$$

Similarly

$$\delta m_2 = \frac{N_2 a_2 + N_2' a_2' + \ldots}{f'(m_2) L_2 a_2},$$

&c. = &c.

It is evident, by the theorem of determinants alluded to in Art. 3, that these are symmetrical expressions.

6. We may conveniently express these results in the form of a rule.

Suppose we have to a first approximation

$$x = M_1 e^{m_1 t} + M_2 e^{m_2 t} + \ldots$$

Eliminate from the differential equations all the variables except x in the usual manner. This may be done by performing on the several equations the operations represented by the minors a, a', &c., $\frac{d}{dt}$ being written for m. Let the equation thus found be

$$f\left(\frac{d}{dt}\right) x = P_1 e^{m_1 t} + P_2 e^{m_2 t} \ldots$$

Then all these terms can be included in the first approximation, provided

$$\delta m_1 = \frac{P_1}{M_1 f'(m_1)}, \qquad \delta m_2 = \frac{P_2}{M_2 f'(m_2)} \text{ \&c.}$$

satisfy the relations

$$p\delta m_1 + q\delta m_2 + \ldots = \delta m_1,$$
$$\text{\&c.} = \text{\&c.}$$

which exist among the roots of the dynamical equation.

7. The general results we have arrived at may be summed up as follows. Though some of the terms of the higher orders may affect the magnitude of the oscillation, yet no term will arise to affect the stability of the motion unless there be some relations between the roots of the dynamical equation of the form

$$pm_1 + qm_2 + \ldots = m_1,$$

where p, q, &c. are all integers. Even if such relations occur, the lowest order of the term is $p+q+\ldots$, and if this be considerable the term will not produce any important effects until a considerable time has elapsed. If a certain relation, just found, hold among these terms, their only effect is slightly to modify the periods of oscillation, without altering the type of motion.

8. *As an example*, let us consider a system of bodies to be oscillating about a position of equilibrium. We know by Lagrange's general solution, that the equation $f(m) = 0$ is of an even order. Its roots are of the form

$$m_1 = \alpha\sqrt{-1}, \quad m_2 = -\alpha\sqrt{-1}, \quad m_3 = \beta\sqrt{-1}, \text{ \&c.}$$

Whatever the numerical values of these may be we have

$$m_1 + m_3 + m_4 = m_1, \qquad m_2 + m_3 + m_4 = m_2, \text{ \&c.}$$

so that the small terms of the third, fifth, &c. orders might affect the stability of the oscillation. But we shall now show that they only affect the periods of oscillation, and not the stability of the system.

Since both sides of Lagrange's equations must be of -2 dimensions in time and the impressed forces are also of -2 dimensions, it is clear that these terms must consist of powers of x, y, &c. $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, &c. and products of an even number of factors of $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c. We know also, by Lagrange's solution, that the co-ordinates take the form

$$x = M_1 \cos(\alpha t + \lambda_1) + \text{\&c.}$$
$$y = M_1' \cos(\alpha t + \lambda_1) + \text{\&c.}$$

The minors a, a', &c. are also all even functions of $\frac{d}{dt}$, hence the equation found after elimination is of the form

$$f\left(\frac{d}{dt}\right)x = P\cos(\alpha t+\lambda) + Q\cos(\beta t+\mu)+\dots$$

Replacing $\alpha\sqrt{-1}$, $-\alpha\sqrt{-1}$, &c. by m_1, m_2, &c. we find by Art. 6,

$$\delta m_1 = \frac{P}{2M_1 f'(m_1)}, \quad \delta m_2 = \frac{P}{2M_1 f'(m_2)}.$$

Since $f'(m)$ is of odd dimensions, and $m_1 = -m_2$ we clearly have $\delta m_1 = -\delta m_2$, and therefore the test is satisfied.

9. [As another example let us apply the rule of Art. 6 to some very simple case which will involve no algebraical substitutions of any length.

The motion of a simple pendulum under the action of gravity may be made to depend on the equation

$$\frac{d^2x}{dt^2} + \alpha^2 x = \beta x^3 \dots\dots\dots\dots (1),$$

where α and β are two constants and x is the inclination of the pendulum to the vertical which is supposed to be small. The first approximation to the motion is

$$x = M_1 e^{m_1 t} + M_2 e^{m_2 t} \dots\dots\dots\dots (2),$$

where m_1 and m_2 are the roots of the equation $m^2 + \alpha^2 = 0$. Our object is to ascertain by help of the rule given in Art. 6 whether the small force represented by Bx^3 renders this first approximation unstable or merely slightly alters the numerical values of m_1 and m_2.

The two roots are connected by the relation

$$m_1 + m_2 = 0 \dots\dots\dots\dots (3).$$

Substituting the value of x on the right-hand side, we have

$$\left(\frac{d^2}{dt^2} + \alpha^2\right)x = 3\beta M_1 M_2 (M_1 e^{m_1 t} + M_2 e^{m_2 t}) + \&c.$$

Hence by the rule in Art. 6

$$\delta m_1 = \frac{3\beta M_1^2 M_2}{2m_1 M_1}, \qquad \delta m_2 = \frac{3\beta M_1 M_2^2}{2m_2 M_2}.$$

These clearly satisfy the relation

$$\delta m_1 + \delta m_2 = 0,$$

and therefore the first approximation taken above, so far as the disturbing force Bx^3 is concerned, is stable.

If the small force had been $\beta\left(\frac{dx}{dt}\right)^3$ instead of βx^3, it is easy to see in the same way that

$$\delta m_1 = \frac{3\beta M_1^2 M_2 m_2^2 m_2}{2m_1 M_1}, \qquad \delta m_2 = \frac{3\beta M_1 M_2^2 m_1 m_2^2}{2m_2 M_2},$$

so that the relation $\delta m_1 + \delta m_2 = 0$ would not have been satisfied. The first approximation taken above is therefore not sufficiently accurate to serve as a first approximation throughout the motion.

In this example we have considered the effect of a small force of the third order in disturbing the stability of the motion given by equation (1). The same equation will obviously occur in many other cases of motion. For example, let a particle describe a circular orbit about a centre of force situated in the centre. If slightly disturbed the equation giving the disturbance x in the radius vector takes the form

$$\frac{d^2x}{dt^2} + \alpha^2 x = \beta x^2 + \gamma x^3 + \dots$$

where α, β and γ are constants. Similar remarks will therefore apply to this case also.]

10. When the coefficients of the equation of motion are not strictly constant, but yet do not vary much, then we may transpose the small variable parts of these terms to the right-hand side of the equation, and treat their products by the differential coefficients of the co-ordinates as small quantities of the second order. Suppose the variable part of one of these coefficients to be $p \sin nt$, where p is small, and let $f(m) = 0$ be the equation giving the periods of oscillation of the system when the coefficients are taken constant. Then it is clear that unless n is nearly equal to the sum or difference of two values of m, this term cannot rise into importance. On proceeding to higher orders we see that these terms cannot produce important effects unless some commensurable relation between m and the roots of the equation $f(m) = 0$ should be very nearly satisfied.

11. It should be remarked that when the coefficients are not constant it is not a sufficient test of stability that they should always satisfy the conditions of stability obtained by giving them their instantaneous constant values. Thus if the equation of motion were

$$\frac{d^2x}{dt^2} + \frac{1}{4t^2}x = 0,$$

the coefficient of x is always positive, yet as the equation is satisfied by $x = a\sqrt{t}$, x may become as great as we please.

Even if the coefficients are nearly constant, we must yet examine, by the rules just given, if their small changes are so timed as gradually to increase the oscillation until the divergence from the given motion is no longer small.

[Suppose a system to be oscillating so that its motion is determined by the equation

$$\frac{d^2x}{dt^2} + qx = 0,$$

where q is a known function of t, which during the time under consideration always lies between β^2 and β'^2 the latter being the greater. Let the system be started with an initial co-ordinate x_0 and an initial velocity x_0' in a direction tending to increase x. It may be shown that the system will begin to return, i.e. x will begin to decrease before x becomes as great as $\sqrt{x_0^2 + \frac{x_0'^2}{\beta^2}}$. If $\pm m$, $\mp m'$ be two successive maximum values of x, we may also show that m' cannot be so great as $\frac{\beta'}{\beta} m$, and that the time from one maximum to the next lies between $\frac{\pi}{\beta}$ and $\frac{\pi}{\beta'}$].

CHAPTER VIII.

The Hamiltonian Characteristic or Principal functions when found determine at once the motion of the system from one given position to another, and whether the motion is stable or unstable. *Arts.* 1—3.

Examples with a mode of effecting the integration $S=\int Ldt$ *in small oscillations.* *Arts.* 4—7.

The Characteristic function supplies the condition that the motion is stable as to space only, while the Principal function gives the conditions that it is stable both as to space and time. *Art.* 8.

In what sense the motion is unstable if either of the two Hamiltonian functions is a minimum. *Arts.* 9—14.

1. If we had any convenient methods of finding the Hamiltonian Characteristic or Principal function, we might determine without difficulty the conditions of stability of a dynamical system at the same time that we deduce the integrals of the equations of motion. But it is very difficult to discover either of these functions by an *à priori* method. We have indeed differential equations which they must satisfy, and Jacobi has taught us what kind of solution will serve our purpose. But the difficulty of finding these solutions is as great as that of solving the equations of motion. For these reasons it does not seem necessary to dwell on the uses of these functions.

2. Suppose the Principal function S of a dynamical system to have been found in terms of the initial co-ordinates θ_0, ϕ_0 and the co-ordinates θ, ϕ and the time t. Let the semi-vis viva be given by

$$T=P\frac{\theta'^2}{2}+Q\theta'\phi'+R\frac{\phi'^2}{2},$$

where P, Q, R are known functions of θ, ϕ. Let P_0, Q_0, R_0 be the values of these when θ_0, ϕ_0 are written for θ, ϕ. The final integrals of the equations of motion are then given by

$$\left.\begin{aligned} -\frac{dS}{d\theta_0} &= P_0\theta_0' + Q_0\phi_0' \\ -\frac{dS}{d\phi_0} &= Q_0\theta_0' + R_0\phi_0' \end{aligned}\right\}.$$

Let the system receive any disturbance at the time $t=0$, so that while starting from the same initial position, its initial velocities are slightly altered. Let x_0', y_0' be these initial changes of θ_0' and ϕ_0' and let $\theta+x$, $\phi+y$ be the co-ordinates of the system at the time t. Then we have

$$\left.\begin{aligned} -\frac{d^2S}{d\theta_0 d\theta}x - \frac{d^2S}{d\theta_0 d\phi}y &= P_0x_0' + Q_0y_0' \\ -\frac{d^2S}{d\phi_0 d\theta}x - \frac{d^2S}{d\phi_0 d\phi}y &= Q_0x_0' + R_0y_0' \end{aligned}\right\}.$$

Here x_0', y_0' are small arbitrary quantities, hence x and y will be small if none of the ratios of $\frac{d^2S}{d\theta_0 d\theta}$, $\frac{d^2S}{d\theta_0 d\phi}$, $\frac{d^2S}{d\phi_0 d\theta}$, or $\frac{d^2S}{d\phi_0 d\phi}$ to the determinant

$$\frac{d^2S}{d\theta_0 d\theta}\frac{d^2S}{d\phi_0 d\phi} - \frac{d^2S}{d\theta_0 d\phi}\frac{d^2S}{d\phi_0 d\theta},$$

be large.

If the initial position as well as the initial motion be altered we may find, by a precisely similar process, the conditions that x and y should be small. If the system have more than two independent motions, we have more than two co-ordinates, but the conditions of stability are found in the same way.

3. If x and y be small throughout the whole motion from the one given position to the other, not only does the system not deviate far from its undisturbed course, but the system at any instant is also very nearly coincident with its undisturbed place at the same time. It is important to notice this, for the word "stability" is sometimes used in a different sense.

This condition of stability may be put under a form in which no reference is made to S. Let u and v be the components of momentum of the system corresponding to the co-ordinates θ, ϕ respectively, *i.e.* let $u=\frac{dT}{d\theta'}$, $v=\frac{dT}{d\phi'}$, and let these be expressed as functions of θ_0, ϕ_0, θ, ϕ and t. Then the preceding equation may be written

$$\left.\begin{aligned} \frac{du}{d\theta_0}x + \frac{dv}{d\theta_0}y = \alpha \\ \frac{du}{d\phi_0}x + \frac{dv}{d\phi_0}y = \beta \end{aligned}\right\},$$

where α and β are small arbitrary quantities. The condition of stability is that the values of x and y thus found should be small.

4. As an example let us consider the case of a projectile. If θ be the horizontal, and ϕ the vertical co-ordinate of the particle, we have

$$\left.\begin{aligned} S &= \frac{(\theta-\theta_0)^2 + (\phi-\phi_0)^2}{2t} - \tfrac{1}{2}gt(\phi+\phi_0) - \tfrac{1}{24}g^2t^3 \\ T &= \tfrac{1}{2}(\theta'^2 + \phi'^2) \end{aligned}\right\}.$$

The equations to find x and y are evidently

$$\left.\begin{aligned} \frac{1}{t}x = x_0' \\ \frac{1}{t}y = y_0' \end{aligned}\right\}.$$

Hence the system continually deviates more and more from its undisturbed place.

5. In order to calculate the form of S when a system is oscillating about a state of motion, it is convenient to choose as co-ordinates some small quantities x, y which vanish in the given state of motion. Let the Lagrangian function be written in the form

$$L = L_0 + L_1 + L_2,$$

where L_n is a homogeneous function of x, y, x', y'. Then by a theorem of Euler's

$$L = L_0 + \Sigma\left(\frac{dL_1}{dx}x + \frac{dL_1}{dx'}x'\right) + \tfrac{1}{2}\Sigma\left(\frac{dL_2}{dx}x + \frac{dL_2}{dx'}x'\right),$$

where the Σ's imply summation for all co-ordinates.

As in Art. 9 of Chap. IV. we have

$$\frac{d}{dt}\frac{dL_1}{dx'} = \frac{dL_1}{dx},$$

and the oscillations are given by

$$\frac{d}{dt}\frac{dL_2}{dx'} = \frac{dL_2}{dx}.$$

Hence we find

$$L = L_0 + \Sigma \left(x \frac{d}{dt} \frac{dL_1}{dx'} + x' \frac{dL_1}{dx'} \right) + \tfrac{1}{2} \Sigma \left(x \frac{d}{dt} \frac{dL_2}{dx'} + x' \frac{dL_2}{dx'} \right).$$

Integrating we have

$$S = \int L_0 dt + \Sigma \left[x \frac{dL_1}{dx'} \right]_0^t + \tfrac{1}{2} \Sigma \left[x \frac{dL_2}{dx'} \right]_0^t.$$

Thus the integration has been effected, but in order to express S as a function of x_0, y_0, x, y and t, it will be necessary to find x_0', y_0', x', y' in terms of these quantities.

6. As an example, let the position of the system depend on one co-ordinate x and let

$$L = L_0 + A_1 x + B_1 x' + \tfrac{1}{2} A_{11} x^2 + \tfrac{1}{2} B_{11} x'^2 + C_{11} x x',$$

where the coefficients are all constants. We then find by the process just indicated that $A_1 = 0$ and

$$S = L_0 t + B_1 (x - x_0) + \frac{C_{11}}{2} (x^2 - x_0^2)$$
$$+ \frac{m B_{11}}{2} \frac{(x^2 + x_0^2)(e^{mt} + e^{-mt}) - 4 x x_0}{e^{mt} - e^{-mt}},$$

where $m^2 = \frac{A_{11}}{B_{11}}$. Applying the criterion of stability we find that $\frac{d^2 S}{dx_0 dx}$ will finally become small if m is real. The motion is therefore unstable or stable according as A_{11}, B_{11} have the same or opposite signs.

7. If the position of the system depend on two co-ordinates x, y, let

$$L = L_0 + A_1 x + A_2 y + B_1 x' + B_2 y'$$
$$+ \tfrac{1}{2} A_{11} x^2 + A_{12} xy + \tfrac{1}{2} A_{22} y^2 + \tfrac{1}{2} B_{11} x'^2 + B_{12} x' y' + \tfrac{1}{2} B_{22} y'^2$$
$$+ C_{11} x x' + C_{12} x y' + C_{21} y x' + C_{22} y y'.$$

We then find

$$S = L_0 t + B_1 (x - x_0) + B_2 (y - y_0)$$
$$+ C_{11} \frac{x^2 - x_0^2}{2} + \frac{C_{12} + C_{21}}{2} (xy - x_0 y_0) + C_{22} \frac{y^2 - y_0^2}{2} + \sigma,$$

where

$$\sigma = \frac{B_{11}}{2} (x x' - x_0 x_0') + \frac{B_{12}}{2} (x y' + x' y - x_0 y_0' - x_0' y_0) + \frac{B_{22}}{2} (y y' - y_0 y_0').$$

If we now express x', y', x_0', y_0' in terms of x, y, x_0, y_0 and t, we find for σ a fraction whose numerator is a homogeneous quadratic function of x, y, x_0, y_0, the coefficients being linear functions of exponentials of t, and whose denominator is another linear function of the same exponentials. These exponentials become sines and cosines when the motion is stable. Thus when the given motion is *steady* the simplest inspection of the form of σ will determine whether the motion is stable or not.

Referring the motion to principal co-ordinates for the sake of brevity, and writing $2G = C_{12} - C_{21}$, we find that σ must satisfy the differential equation

$$\frac{d\sigma}{dt} + \frac{1}{2B_{11}}\left(\frac{d\sigma}{dx} + Gy\right)^2 + \frac{1}{2B_{22}}\left(\frac{d\sigma}{dy} - Gx\right)^2 = \tfrac{1}{2}A_{11}x^2 + \tfrac{1}{2}A_{22}y^2.$$

This equation is obviously satisfied by such a function as that just described. The solution of this equation may be reduced to linear equations and thus σ may be found. But it is unnecessary to dwell on this, for this would be equivalent to returning to the Hamiltonian equations.

8. If we wish to determine the condition that the general course of a dynamical system is stable without requiring it should be near its undisturbed place at any the same time, it is more convenient to use the Characteristic function. Suppose that the Characteristic function has in Jacobi's manner been expressed as a function of the co-ordinates θ, ϕ, the constant h of vis viva and two arbitrary constants a_1, a_2. Then

$$V = f(\theta, \phi, h, a_1) + a_2.$$

The relation between θ and ϕ, which may be called the equation to the path of the system, is given by

$$\frac{dV}{da_1} = b_1,$$

where b_1 is another constant. Let the system be disturbed from the same initial position so that the whole energy is unaltered. The change in ϕ corresponding to any given value of θ is found from

$$\frac{d^2V}{da_1^2}\delta a_1 + \frac{d^2V}{da_1 d\phi}\delta\phi = \delta b_1.$$

Let A be the *initial* value of $\dfrac{d^2V}{da_1^2}$, then

$$\delta\phi = -\frac{\dfrac{d^2V}{da_1^2} - A}{\dfrac{d^2V}{da_1 d\phi}}\cdot\delta a_1.$$

The condition that the path should be stable is that the coefficient of δa_1 should not be large.

We might also use the function called Q by Sir W. R. Hamilton, but it seems unnecessary to dwell more on this subject.

9. The instability of a system may be deduced from the Hamiltonian Principal or Characteristic functions, expressed as a minimum. Suppose a dynamical system to move from one position A to another B in a time t, then the motion may be found by making the first variation of $S=\int_0^t L\,dt$ equal to zero, the time of transit being constant. The constants of integration are determined by the conditions that the co-ordinates have given values when $t=0$ and $t=t$. To determine whether S is a maximum or minimum or neither we must examine the second variation and here we have the assistance of Jacobi's rule. The determination of the constants will depend on the solution of equations and may lead to several different kinds of motion from A to B. One of these will be the actual motion. Let us move B along this until one of the other motions coincides or as we may say approaches indefinitely near to this actual motion. We have then reached a boundary beyond which the integration must not extend if S is to be a maximum or minimum. See Todhunter's *History of the Calculus of Variations*, page 251. Further $\frac{d^2L}{d\theta'^2}$, if θ be a co-ordinate, is positive throughout the limits of integration, so that S will be a minimum and not a maximum.

10. When there are several co-ordinates θ, ϕ, &c. which are to be found as functions of the time, we may easily show that Jacobi's condition is a necessary one, and this is all that we require for the next proposition. If the system can move in two ways from A to B, then $\delta S=0$ along each, and therefore when these two are adjacent we have both $\delta S=0$ and $\delta(S+\delta S)=0$. This shows that the second variation *can be made* to vanish by taking one variation through the other. This second variation will then be the same as the quadratic term of the series obtained by changing the co-ordinates θ, ϕ into $\theta+\delta\theta$, $\phi+\delta\phi$, because we can take $\delta^2\theta=0$ and $\delta^2\phi=0$. Hence as the sum of the terms of the third order does not, in general, vanish for this displacement, it is clear that S cannot be either a maximum or a minimum.

Let the actual motion be from A to B, and let a neighbouring motion starting from A lead the system to a position C reached in the same time along the actual motion before reaching B. Then we can show that a variation of the actual motion from A to B can be found which makes $\delta^2 S$ of any sign. Let P be any position on the neighbouring motion before reaching C, and Q, one on the actual motion after passing C. Then considering P and Q as fixed and also the time of transit, the motion along PCQ cannot make $\delta S = 0$; for this condition is known to lead to the ordinary dynamical equations, and it is clear that (impulsive forces being set aside) no actual motion can be discontinuous. But there is discontinuity at C, for otherwise when the system is started from B towards A, two courses would be open to the system on arriving at C. Hence the first variations of S for an imaginary motion along PCQ are not zero, and therefore may have any sign. But since the discontinuity at C is of the first order of small quantities, this first variation is of the second order. Now the value of S for the actual motion is equal to that along the neighbouring motion to C and then along the actual motion to B. Hence, P and Q being still fixed, variations of the actual motion from A to B can be found which make $\delta^2 S$ of any sign.

11. Let us apply this theory to determine the stability of a given state of motion. First let us suppose the given motion to be steady and to depend on only two variables. If we use the function S there will be one co-ordinate and the time, if V two co-ordinates. Let the system be disturbed at any moment by an alteration of the velocities of its several parts, so that the initial position of the disturbed motion is an undisturbed position. If the motion be stable the system will oscillate about the undisturbed motion, the oscillation repeating itself at a constant interval. It follows therefore by Jacobi's rule that S or V cannot be a minimum for a period longer than the time of a half-oscillation. If therefore S or V be a minimum for all variations, starting from A and ending at B, where B is a position on the steady motion reached by the system at an interval as long as we please, then the motion is unstable.

If we give a meaning to the word "stable" somewhat different* from its usual signification, we may extend this proposition to determine a test of the stability of any motion, whether steady or not. All we have assumed is, that, if the motion be not altogether unstable, there are *some* disturbances which will cause the system periodically to assume the same positions as it would have done if it had been undisturbed, but the interval of these periods may be any whatever provided the first be finite. If we use the Characteristic function, these disturbances must be such as not to alter the constant of vis viva, and if the Principal function, they must be such as to bring the system to an undisturbed position in the same time.

* This meaning does not always agree with the results of Art. 14, Chap. IV. [See also Arts. 17 and 18, Chap. IV.]

12. Next let us suppose that as the system proceeds from the initial position A along the actual motion, S ceases to be a minimum at some position B. The conditions for a minimum are of two kinds. Suppose the system to depend on two co-ordinates θ, ϕ, and let L be the Lagrangian function, then (1) we must have $\frac{d^2L}{d\theta'^2}$ and $\frac{d^2L}{d\theta'^2}\frac{d^2L}{d\phi'^2}-\frac{d^2L}{d\theta' d\phi'}$ both positive, and (2) it must be possible to choose three arbitrary constants which enter into a very complicated expression, so that this expression may never become infinite between the limits of integration. The first condition is clearly always satisfied since the vis viva of any system is necessarily positive for all values of θ' and ϕ'. The second condition will fail if there are two neighbouring motions by which the system can proceed from A to any position between A and B. If this be the mode of failure, it is clear from the reasoning of Art. 2 that the conditions of stability are satisfied for one kind of disturbance, and that therefore some at least of the harmonic motions are stable or oscillatory, though the motion may be unstable for a different kind of disturbance.

13. [These conditions become much simpler when the position of the system is determined by one co-ordinate, or when the Lagrangian function can be reduced to depend on one co-ordinate. Let this co-ordinate be so chosen that it vanishes along the given motion, and let us also suppose that both it and its differential coefficient with regard to t, are small for all neighbouring constrained motions. Let this co-ordinate be called θ and let the Lagrangian function be

$$L = L_0 + A_1\theta + B_1\theta' + \frac{1}{2}A_{11}\theta^2 + \frac{1}{2}B_{11}\theta'^2 + C_{11}\theta\theta'.$$

Then since the Lagrangian equation of motion is satisfied by hypothesis when $\theta = 0$, we have $A_1 = B_1'$ where the accent, as usual, denotes differentiation with regard to t.

If the system be now conducted from the initial position A to any other position B, both on the given motion, by any neighbouring mode of motion, we have

$$S = \int L_0 dt + \int\left(\frac{1}{2}B_{11}\theta'^2 + C_{11}\theta\theta' + \frac{1}{2}A_{11}\theta^2\right)dt.$$

If $\theta = u$ be any solution of the Lagrangian equation

$$\frac{d}{dt}(B_{11}\theta' + C_{11}\theta) = C_{11}\theta' + A_{11}\theta,$$

we may write the function S in the form*

$$S = \int L_0 dt + \frac{1}{2}\int B_{11}\left(u\frac{d}{dt}\frac{\theta}{u}\right)^2 dt.$$

The second term is essentially positive, since B_{11} must be positive. Hence S is a minimum along the given motion unless we can so choose the arbitrary displacement θ as to make $\frac{d}{dt}\frac{\theta}{u} = 0$. This gives $\theta = cu$ where c is some constant. But θ must vanish at the two limits A and B, hence this choice of θ is excluded unless there is some neighbouring mode of motion by which the system could move freely from the given initial position A to the position B. The result is that S cannot cease to be a minimum before the *first* instant at which some neighbouring motion will bring the system (starting from A) into coincidence with some contemporaneous position on the given motion. If the given motion be steady, it follows that S cannot cease to be a minimum before a time which is half that of a complete oscillation.

We thus have a test of stability. If the system depend on one co-ordinate and *if* S *be a minimum when the limits of integration are from the initial position* A *to all positions on the actual motion, that motion is unstable. But if* S *cease to be a minimum at some point* C, *then the actual motion is stable from* A *to* C.]

* [Following Lagrange's rule we may write the second term of S in the form

$$-\lambda\theta + \int\left\{\frac{1}{2}B_{11}\theta'^2 + (C_{11}+2\lambda)\theta\theta' + \left(\frac{1}{2}A_{11}+\lambda'\right)\theta^2\right\}dt.$$

The quantity outside the integral sign is to be taken between the given limits and is zero, since θ vanishes at each limit. Let us now put

$$\frac{C_{11}+2\lambda}{B_{11}} = -\frac{u'}{u}.$$

It is clear that this value of λ cannot be infinite between the limits of integration unless u vanishes. For by hypothesis u and u' are both finite and the coefficients B_{11} and C_{11} in the Lagrangian function are also finite. It then easily follows from the equation

$$\frac{d}{dt}(B_{11}u + C_{11}u) = C_{11}u + A_{11}u,$$

that
$$(C_{11}+2\lambda)^2 = B_{11}(A_{11}+2\lambda').$$

Hence the second term of S becomes

$$\frac{1}{2}\int B_{11}\left(\theta' - \theta\frac{u'}{u}\right)^2 dt,$$

which is the result in the text. This might also have been deduced from Jacobi's general transformation with one independent variable given in Prof. Jellett's *Calculus of Variations* or Prof. Price's *Differential Calculus*.

If $u=0$ between the limits of integration this transformation fails, but, as is evident from the argument in the text, we choose $\theta=u$ to represent a neighbouring free motion such that $u=0$ just before the system reaches A; also B is so placed that the next instant at which $u=0$ is after the system has passed B.]

14. [We shall conclude the chapter with the application of this criterion of stability to some simple case.

A particle describes a circular orbit about a centre of force situated in the centre. It is required to deduce the conditions of stability as to space from the Characteristic function V.

Let a be the radius of the circle, n the angular velocity of the particle about the centre O. Let $\phi(a)$ be the law of force. Let A and B be two points taken on the circular orbit, and let the particle be conducted from A to B by some neighbouring path with the same energy as in the circular orbit. Let $r = a + \rho$ be the radius vector of this path, corresponding to any angle θ.

If v be the velocity at any point of this path, we easily find

$$v = an\left\{1 - \frac{\rho}{a} - \left(\frac{a\phi'(a)}{\phi(a)} + 1\right)\frac{\rho^2}{2a^2}\right\}.$$

If s be the arc of the path, we have

$$\frac{ds}{d\theta} = a + \rho + \frac{1}{2a}\left(\frac{d\rho}{d\theta}\right)^2.$$

If the angle $AOB = \beta$, we therefore have

$$V = \int v ds = a^2 n\beta + \frac{n}{2}\int_0^\beta \left\{\left(\frac{d\rho}{d\theta}\right)^2 - p^2\rho^2\right\} d\theta,$$

where $p^2 = \dfrac{a\phi'(a)}{\phi(a)} + 3$.

If the neighbouring path be a free path described with the same energy, its equation is

$$\rho = L \sin p\theta,$$

where θ is measured from the radius vector OA, and L is an arbitrary constant. This free path will cut the circle again in some point C. If the angle $AOC = \gamma$, we have $p\gamma = \pi$.

If B coincide with C, we find by substituting this value of ρ in the expression for V, that the second term of V is zero. If B be beyond C but such that the angle COB is less than γ, draw two free paths one from A as before and the other backwards from B to meet the former in some point P. Then the angle $AOP = \frac{1}{2}\beta$. If the particle be conducted from A to P along one path and from P to B along the other, we find that the excess of the action over the action in the circular arc AB is equal to

$$\frac{np}{2} L^2 \sin p\beta.$$

Since $p\beta$ is greater than π and less than 2π, this excess is negative. The action along the circular arc is therefore not a minimum if B be beyond the first intersection of a neighbouring free path.

Lastly we may show that the action is a minimum if B lie between A and C. To prove this we write the integral in the expression for V in the form

$$\Big[-\lambda\rho^2\Big]_0^\beta + \int_0^\beta \left\{\left(\frac{d\rho}{d\theta}\right)^2 + 2\lambda\rho\frac{d\rho}{d\theta} + \left(\frac{d\lambda}{d\theta} - p^2\right)\rho^2\right\} d\theta.$$

The first of these two terms is zero since ρ vanishes at each limit. Following Lagrange's rule we make

$$\frac{d\lambda}{d\theta} - p^2 = \lambda^2.$$

The integral then becomes

$$\int_0^\beta \left(\frac{d\rho}{d\theta} + \lambda\rho\right)^2 d\theta.$$

This is always positive and the action along the circular arc is a minimum. The argument however requires that λ should not become infinite between the limits of integration. It is easy to see that

$$\lambda = p\tan(p\theta + E),$$

where E is a constant to be chosen at our pleasure. But if β exceed $\frac{\pi}{p}$ it is impossible to choose E so as to keep λ finite between the limits of integration. Hence the action is a minimum only if the angle AOB subtended by the limiting positions at the centre is less than $\frac{\pi}{p}$. The circular orbit is therefore stable if p^2 be positive.

If, however, p^2 be negative, the expression for λ changes its character. Writing $-q^2$ for p^2, we find

$$\frac{q+\lambda}{q-\lambda} = Ee^{-2q\theta},$$

thus the value of λ can be chosen so as not to be infinite for all positive values of θ. In this case the function V is a minimum for all arcs AB however distant B may be from A. The circular motion is therefore unstable.]

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