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SUMMARY

Fault monitoring schemes are proposed and evaluated for the class of systems described by linear, stochastic difference equations. The systems are subject to sudden faults or changes in their parameter values, which if not detected and compensated for, will cause degradation to the performance of the Kalman filter used to estimate the state of the noisy system. The monitoring scheme is required to perform detection, isolation, identification of time and size of fault and subsequent post-fault reorganization.

The types of faults considered are represented by a term which is additional to the failed parameter in the system model. The parameters under consideration are the means and variances of the state and measurement noise sequences and the coefficients in the state and measurement equation.

On the assumption that system faults will occur infrequently, the fault monitoring scheme utilises information supplied by a Kalman filter based on the assumption of no faults. This information, namely the filter innovations sequence and its joint probability density function, is then used by a secondary monitoring system which performs a statistical analysis on the innovations to decide if a fault has occurred.

Several structures for the monitoring scheme are proposed, ranging from simple statistical tests to more complicated generalised likelihood ratio tests. A simple logic, based on the different effects the various types of faults have on the joint probability distribution of the innovations, is developed which permits detection and partial isolation to be performed using the simpler statistical tests, while fault identification is achieved by the subsequent use of generalised likelihood ratio tests. Alternatively, the two stages may be implemented separately,

where specific application requirements so dictate.

The analysis is verified by digital computer simulation of a first order system, and the proposed monitoring schemes are shown to operate successfully under well defined conditions.

Possible generalisations and extensions are also discussed.

MEMORANDUM

The accompanying dissertation is based on work carried out by the author at Brunel University between October 1976 and March 1980.

All work and ideas in this dissertation are original, unless otherwise acknowledged in the text or by references. The work has not been submitted for another Degree in this University, nor for the award of a Degree or Diploma at any other Institution.

The main contributions the author claims to have made to the subject of fault monitoring schemes for linear stochastic systems are:

1. The development of mathematical models for linear stochastic systems subject to occasional changes in the values of the noise and dynamical parameters.

2. The development of equations, in recursive form, modelling the effect of the various types of faults on the innovations sequence of a Kalman filter designed to estimate the state of the stochastic system under no-fault conditions.

3. The proposal of appropriate statistical methods for the monitoring of the innovations sequence and the identification of the condition of the system using hypothesis-testing techniques.

4. The suggestions for system reorganization following a fault occurrence.

ACKNOWLEDGMENTS

I would like to thank my supervisor Dr. J. F. Marsh for his constant help through useful suggestions and encouragement during the entire period of this work.

I would also like to express my appreciation for the use of facilities in the Department of Electrical Engineering and Electronics at Brunel University and for the help given by the Computing Centre in carrying out the computer simulation tests.

My thanks also to Prof. D. R. Cox and Dr. J. C. Allwright of Imperial College for some inspiring comments in their relevant fields.

My thanks are also due to the Admiralty Compass Observatory, Ditton Park, for their partial support of this work through a grant.

Finally, my deepest gratitude to my parents who have patiently and wholeheartedly supported my studies over the past twenty-three years.

LIST OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF PRINCIPAL SYMBOLS

$\|\|G\|\|$	norm of matrix G (Euclidean)
$\|G\|$	determinant of matrix G
G^T	transpose of matrix G
G^{-1}	inverse of matrix G
$\hat{s}$	optimum value of s
$\{y(k)\}$	sequence of y(k)'s
$N(m,s^2)$	normal distribution of mean m, variance s^2
$>>$	much greater than
$\bar{x}$	mean value of x
$\hat{\bar{x}}$	sample mean value of x
N^+	set of positive integers
R^n	set of n-dimensional reals

Section numbers are referred to in the text without parentheses in contrast to equation numbers which are quoted in parentheses.

Small Greek letters without arguments generally denote scalar constants. Small Greek or English letters with arguments (usually time) denote scalar variables (with the exception of the Kalman gain, which is written with a capital K in the scalar case to avoid confusion with the time index, k). Capital English letters will usually denote matrices.

Suffices a, b, c, d, e, f denote that the subscripted variable is relevant to fault a, b, c, d, e, f respectively.

CHAPTER I: INTRODUCTION

I.1 Background to the problem of fault monitoring.

The field of fault monitoring in control systems has received much attention during the last decade.

Malfunctions of plant equipment and instrumentation often have an adverse effect on factors such as performance, safety, profitability and security and the study of aspects of reliability and redundancy in dynamical systems is now established as a separate branch of Control Theory. This area of investigation may be termed automatic diagnostics, leading to automatic alarm annunciation and post-fault system reorganization.

Implementation of theoretical results in this field has been encouraged by recent developments in low cost, high reliability, low volume computer systems. As new microcomputer configurations become available for on-line control application, the implementation of algorithms, which are highly complex due to the nature of the problems to which they relate, becomes more attractive.

The feasibility and degree of complexity of schemes for fault monitoring depends on the nature of the fault. Complete malfunction of equipment is often relatively straightforward to detect, but detection of process degradation, with which the present work is concerned, often presents a more complex problem.

I.2 Previous work on fault monitoring.

Research in fault monitoring is relatively new and deals mainly with fault detection. Thus the problems of fault isolation and estimation are treated by a relatively few authors. Consequently the problem of system reorganization is also not researched.

The methods used can be classified in various ways according to the adopted system model, filter type and detection-isolation mechanism. Of particular mention are:

Voting techniques, based on systems that possess a high degree of parallel hardware redundancy. Simple logic is then developed to detect faults and eliminate faulty instruments *[29]*. Voting systems are in general relatively easy to implement and often provide fast detection of faults of large magnitude but give rise to difficulties in detecting faults of small magnitude (sometimes referred to as "hard" and "soft" faults respectively). This technique does not provide fault isolation or estimation.

Multiple hypothesis filter-detectors: A large class of adaptive estimation and fault detection schemes involves the use of a series of parallel linear filters based on different hypotheses concerning the underlying system behaviour *[4]*, *[6]*, *[7]*, *[32]*, *[33]*, *[42]*. Filters for each of the models are constructed and the innovations from the various filters are used to compute the conditional probability that each system model is the correct one. In this manner, if the system parameters can be constrained to take a discrete set of values, simultaneous system identification, state estimation and fault detection can be performed. This technique, however, yields suboptimal filter

estimates when no faults are present in the system.

A modification of the multiple hypothesis technique involves the use of the sequential probability ratio test (SPRT) in detecting a switch between different models *[62]*. The SPRT test proceeds as follows: given two hypotheses concerning the system behaviour, the a-posteriori probabilities of the two models are computed and the logarithm of their ratio compared to two thresholds. If it exceeds one threshold or falls below the other, the test is terminated with a decision corresponding to the crossed threshold; otherwise decision is deferred. The SPRT is a powerful test with various optimum properties, but it can only be used when all parameters under both hypotheses are completely known.

Jump process formulations: The nature of the problem of fault detection suggests the use of jump processes in devising system design methodologies. In such formulations an a-priori distribution characterises the potential faults which are modelled as jumps. The size of the possible faults is usually assumed known, or modelled as a random variable. Jump process formulations appear to be quite natural for fault detection problems. However, approximations have to be made in the analysis in order to obtain implementable solutions which in turn impose limitations on the capabilities of the designs *[34]*.

Failure-sensitive filters: Several methods have been developed for the design of filters that are sensitive to specific faults. One method involves the inclusion of several "failure-states" in the dynamical system model *[64]*. If filter estimates of the fault states differ markedly from their nominal values, a fault is declared. This approach provides fault isolation and estimation at the expense of increased dimensionality and some performance degradation under no fault conditions.

An alternative to the addition of failure-states is a class of

detector filters designed so as to highlight certain faults in the filter residuals *[28]*. This methodology is extremely useful conceptually, can be used to detect a wide variety of faults and provides detailed fault isolation information. It is however suboptimal as an estimator but may be used as an auxiliary monitoring system. The major limitation is its applicability to time-invariant systems only.

Innovations-based detection systems: These methods involve the monitoring of the innovations or residuals of the filter based on the hypothesis of normal system operation. In such a configuration the overall system uses the normal filter until the innovations monitoring system detects some sort of abnormal behaviour. The fact that the monitoring system can be attached to a filter-controller feedback system is particularly appealing since overall system performance is not degraded until after the monitor signals a fault and since the monitoring system can be designed to be added to an existing system.

A number of possible statistical tests can be performed on the innovations, thus yielding information concerning the occurrence or otherwise of a fault *[32]*, *[33]*, *[35]*. This is basically a detection method.

Finally, the generalised likelihood ratio (GLR) technique, motivated in part by the shortcomings of the simpler innovations-based fault detection techniques, can be applied to a wide range of faults making use of the knowledge of the different effects such faults have on the filter innovations. The method provides an optimum decision rule for fault detection and provides useful fault identification information for use in system reorganization subsequent to the detection of a fault. In addition a number of simplifications can be devised and an analytical study of the tradeoff between complexity and performance can be carried out in simple cases *[36]*.

Succesful results of the GLR method have been reported in the case of jump biases in the state of a dynamical system *[43]*.

In the present work, the fault monitoring schemes which are proposed adopt the innovations based GLR technique but extend its applicability to fault types not so far researched. These include step biases and parametric faults in both the state and measurement equations. Simpler statistical tests for these faults are also proposed.

I.3 Statement of the problem.

The problems to be investigated are principally concerned with the design of fault monitoring schemes for the class of scalar, time invariant dynamical systems described by the following set of linear stochastic difference equations:

$$x(k+1)=\phi x(k)+w(k) \qquad (I.1)$$

$$y(k)=\eta x(k)+v(k) \qquad (I.2)$$

where $x(k)$ is the system state at time t_k; $\{x(k)\}$, $k=0,1,\ldots$ is therefore the state sequence; ϕ is a finite non-zero, non-random transition constant; $\{y(k)\}$, $k=1,2,\ldots$ is the measurement or observation sequence; η is a finite non-zero, non-random constant coefficient; $\{w(k)\}$, $k=0,1,\ldots$, $\{v(k)\}$, $k=1,2,\ldots$ are stationary, white gaussian sequences, with

$$w(k)\sim N[0,q] \; ; \; q>0$$

$$v(k)\sim N[0,r] \; ; \; r>0$$

In addition, the initial state x(0) is usually considered gaussian with,

$$x(0) \sim N[\bar{x}(0), p(0)]$$

and x(0), w(k) and v(k) are assumed mutually independent for all k.

Equation (I.1) is termed the "state" or "plant" equation whilst equation (I.2) is called the "measurement" or "observation" equation. These equations form the model for the discrete state estimation function which proceeds on receipt of the measurement sequence {y(k)} given a set of parametric data. No loss of generality results from the absence of a deterministic control in equation (I.1), since the error in system state estimation is independent of such an input [40].

The model so defined is not a unique representation of stochastic linear system behaviour. In general, appropriate state or measurement models may be either continuous or discrete for specific applications. Linear systems which are continuous in the state, however, may, after discretization, be represented by equation (I.1), and the discrete modelling of measurements received at an approximately constant update rate is appropriate to a wide range of practical situations.

Although the fault monitoring schemes to be described are principally related to the scalar model, extensions to the general multivariable case are discussed in section III.2.8. The treatment of systems with time-varying parameters is also possible.

Equations (I.1) and (I.2) are illustrated in block diagram form in fig. 1(a).

The noisy measurement sequence {y(k)} is the sole source of information from the actual system regarding the system state x(k). Since accurate knowledge of x(k) is a prerequisite for precise system operation through feedback controls, a state estimator which operates

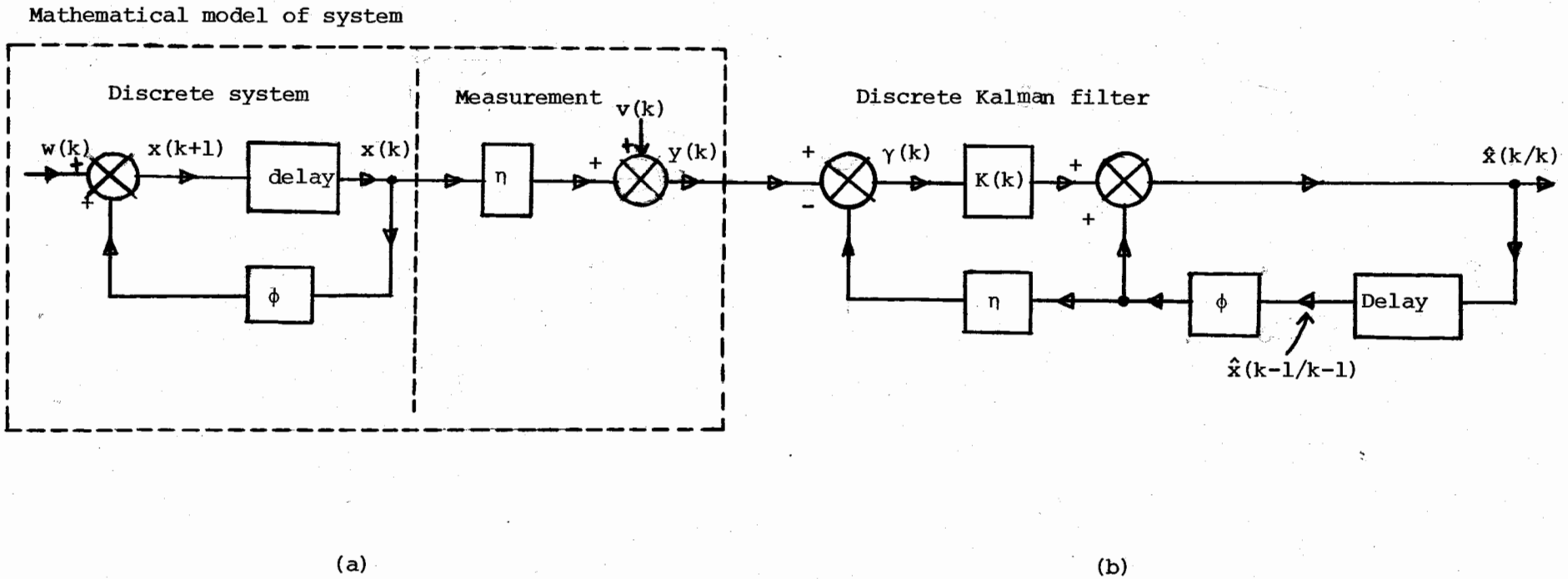

Fig. 1 System model and discrete Kalman filter

on the measurement sequence is often used in practical applications. It may be shown [11] that the estimator which minimises a wide class of loss functions for the system described by (I.1) and (I.2), is the discrete Kalman filter estimator defined by the set of difference equations,

$$\hat{x}(k/k-1) = \phi\hat{x}(k-1/k-1) \tag{I.3}$$

$$p(k/k-1) = \phi^2\, p(k-1/k-1) + q \tag{I.4}$$

$$K(k) = p(k/k-1)\eta[\eta^2 p(k/k-1)+r]^{-1} \tag{I.5}$$

$$\gamma(k) = y(k) - \eta\hat{x}(k/k-1) \tag{I.6}$$

$$\hat{x}(k/k) = \hat{x}(k/k-1) + K(k)\gamma(k) \tag{I.7}$$

$$p(k/k) = (1-K(k)\eta)p(k/k-1) \tag{I.8}$$

where $\hat{x}(k/k-1)$ denotes the optimum estimate of $x(k)$ based on the measurement sequence,

$$y^{k-1} \triangleq \{y(i)\}, i=1, 2, \ldots, (k-1) \tag{I.9}$$

$p(k/k)$ is the variance of the estimation error $x(k)-\hat{x}(k/k)$; $K(k)$ is termed the Kalman gain; $\gamma(k)$ is termed the measurement residual or innovations process. The Kalman filter algorithm is of predictor-corrector form, where equations (I.3) and (I.4) represent the predicted estimate together with its variance and equations (I.7) and (I.8) represent the filter output based on the gain of equation (I.5) together with a correction in the light of new measurement information, given by (I.6). The algorithm is initialised by,

$$\hat{x}(0/0) = \bar{x}(0) \tag{I.10}$$

and

$$p(0/0) = p(0) \tag{I.11}$$

The filter update based on measurements received is illustrated in fig. 1(b).

In this case, where the system parameters are time-invariant, the

propagation of the variance may be conveniently computed a-priori through to a steady state, since equations (I.4), (I.5) and (I.8) are independent of y^k.

It is well known (e.g. [40]) that the Kalman filter estimate is unbiased minimum variance in the case of gaussian disturbances and linear unbiased minimum variance for the general case without the assumption of gaussianness. It then follows that the estimate $\hat{x}(k/k)$ is also the mean of the state $x(k)$ conditioned on the measurement sequence y^k, regardless of the properties of the conditional density function. Thus,

$$\hat{x}(k/k) = E[x(k) \,|\, y^k] \tag{I.12}$$

The unbiasedness property then gives,

$$E[\hat{x}(k/k)] = E[x(k)] \tag{I.13}$$

A change in the assumed value of a parameter will be considered a fault. Faults are assumed to be additive and of step-type and may occur in any of the system noise or dynamical parameters. No a-priori information is assumed on the probability of occurrence of any specific fault at any time. Specifically, the following cases of faults are considered:

(a). Change in the state noise mean.

(c). Additional plant noise.

(d). Change in the measurement noise mean.

(f). Additional measurement noise.

Results are also obtained in the following cases:

(b). Change in ϕ.

(e). Change in η.

The following assumptions on the system model are made:

1. The system is stable, i.e. $|\phi|<1$

2. The system is uniformly completely controllable and uniformly completely observable. This ensures the uniform asymptotic stability of the Kalman filter.

The fault monitoring scheme is required to perform the following operations:

1. Detection of fault
2. Isolation of fault
3. Estimation of time of occurrence and magnitude of fault.
4. Reorganization of system model following a fault.

The faults are assumed to be *single*, i.e. no more than one fault may occur at any time.

I.4 Outline of proposed method.

The proposed fault monitoring schemes make extensive use of statistical signal processing techniques which monitor the received Kalman filter innovations. It is a well known fact that the Kalman filter innovations are especially suited for statistical hypothesis testing since they are uncorrelated, have zero mean and their variance is conveniently computed as part of the filter algorithm.

Each of the faults (a)-(f) will produce a different sequence of residuals, therefore knowledge of their effects can be used to detect, isolate and identify any fault.

It is shown that the innovations sequence may be written as the sum of two distinct terms, as:

$$\gamma(k) = \gamma_0(k) + g_i(k,\theta,\Delta p)$$

where $\{\gamma_0(k)\}$ is the residual sequence which would be obtained if no fault occurred and $g_i(k,\theta,\Delta p)$ is an additive term which depends on the

particular fault i, present time k, time of fault occurrence θ and fault size Δp.

The terms g_i are calculated for each fault (a)-(f) and subsequently used to establish the form of the probability density function (pdf) under alternative hypotheses of a fault occurrence of size Δp at time θ.

The effect of each fault on the statistical properties of the innovations sequence is used to classify the faults and hence devise simple statistical tests which can be used to detect and partially isolate the fault to a class which contains at most two probable cases. The tests used are: the sign test for the innovations mean and a sample variance and first order serial correlation test for the whiteness property.

The particular form of the joint pdf of the innovations sequence, which depends on the occurred fault, can be used in more sophisticated statistical decision algorithms which perform fault detection, isolation and identification. The statistical method used is the generalised likelihood ratio (GLR) test which involves calculating the ratio of the maximum likelihood (ML) pdf under each hypothesis and comparing it to a threshold. A decision is then made according to whether the GLR value exceeds the threshold value or not.

The GLR procedure can be used, at least in theory, in systems where the number of possible faults exceeds two. In these cases the GLR statistic has to be calculated for each pair of hypotheses of fault against no fault and the most likely event chosen.

Alternatively, the GLR algorithm can be used to monitor for any single fault or it can be used in conjunction with the simpler detection partial isolation algorithms to monitor any fault in the assumed noise parameters without the need for excessive calculations.

I.5 Chapter organization.

The subsequent work is presented in five chapters.

In Chapter II some well known properties of Kalman filters and dynamical systems in general are reviewed and mathematical models for the class of faults considered are proposed. System models which might experience a fault in their parameters are then constructed using the fault models.

Chapter III deals with the determination of the different effect of each type of fault on the filter innovations. These results are then used to obtain the joint pdf of the innovations sequence in the event of a fault occurrence.

In Chapter IV the additive effect of the faults on the filter innovations and the knowledge of the exact form of the joint pdf of the innovations following a fault, is used to formulate the problem of fault monitoring as a problem in hypothesis testing. Simple statistical tests based on the properties of the filter innovations in normal operation are presented, which perform the functions of fault detection and partial isolation. GLR test procedures are also proposed and their implementation problems examined.

Chapter V contains simulated Monte-Carlo trials using the methods developed in Chapter IV. The tests are classified and conclusions for each case presented.

Chapter VI contains general conclusions and remarks as well as recommendations for future research.

Appendices of mathematical proofs and computer programs are included at the end of the thesis.

CHAPTER II: *MATHEMATICAL FORMULATION*

II.1 Properties of Kalman filter innovations.

In Section I.3 the residual or innovations sequence for the Kalman filter is defined as:

$$\gamma(k) = y(k) - \eta\hat{x}(k/k-1) \qquad (I.6)$$

In any practical scheme for state estimation, it is necessary to track the performance of the filter, in particular to check whether the algorithm is operating on the basis of correct model parameters. Errors in assumed parameter values may lead to the problem of filter divergence *[40]*. In principle, a self-checking procedure based on new information provided by $y(k)$ and the output of the Kalman filter at instant t_k is not possible, however, since all the information in $\{y(k)\}$ is apparently utilised in the generation of $\hat{x}(k/k)$. One such measure, however, is provided by the innovations process, since $\hat{x}(k/k-1)$ is computed in the absence of the new measurement update $y(k)$, and the product $\eta\hat{x}(k/k-1)$ is then interpreted as an estimate of $y(k)$.

Since $\gamma(k)$ is derived from a linear transformation on stochastic processes with known probability laws, $p[\gamma(k)]$ may be defined. It is shown in *[39]* that for a Kalman filter operating with correctly identified parameters, the residual sequence $\{\gamma(k)\}$, $k=1,2,\ldots$ is gaussian, with:

$$\bar{\gamma}(k) \triangleq E[\gamma(k)] = 0; \quad \text{all } k \qquad (II.1)$$

$$c(k,m) \triangleq E[\gamma(k)\gamma(m)] = 0; \quad \text{all } k \neq m \qquad (II.2)$$

$$c(k,k) \triangleq E[\gamma^2(k)] = \eta^2 p(k/k-1) + r; \quad \text{all } k \qquad (II.3)$$

These important properties, valid in the case of correctly identified parameters for equations (I.1) and (I.2), will form the basis of the fault monitoring scheme to be described in Chapter III.

II.2 Stochastic controllability and observability.

In deterministic dynamical systems the concepts of controllability and observability play a very important role in characterising possible system behaviour.

The concept of *controllability* refers to the general property of being able to transfer a system from any given state to any other by means of a suitable choice of control functions. Closely linked to the idea of controllability is that of *observability* which in general means that it is possible to determine the state of a system by measuring only its output, given that the control inputs to the system are known.

The corresponding concepts of observability and controllability of stochastic systems exist and are intimately connected with convergence questions of a-posteriori pdf's such as $p(x(k)\,|\,y^k)$, the conditional density of the state $x(k)$ given y^k.

Consider the time-varying multivariable system corresponding to (I.1)-(I.2) given by the following pair of vector difference equations:

$$\underline{x}(k+1) = \Phi(k+1,k)\underline{x}(k) + \Gamma(k)\underline{w}(k) \qquad \text{(II.4)}$$

$$\underline{y}(k) = H(k)\underline{x}(k) + \underline{v}(k) \qquad \text{(II.5)}$$

where $\Phi(k+1,k)$, $\Gamma(k)$, $H(k)$ are matrices of appropriate dimensions and $\underline{w}(k)$, $\underline{v}(k)$ are vector gaussian random sequences of zero mean and covariance matrices $Q(k)$, $R(k)$ respectively, uncorrelated for all k.

The only control in (II.4) is the random term $\underline{w}(k)$. Thus it seems that if (Φ,Γ) represents a controllable pair $\underline{x}(k)$ should "go everywhere". The last statement can be expressed formally as follows: if N is a set

having positive volume and located anywhere in R^n, then there is strictly positive probability that $\underline{x}(k)$ is in N. To obtain conditions on controllability, consider the pdf of $\underline{x}(k)$:

$$p(\underline{x}(k)) = (2\pi|P(k)|)^{-n/2} \exp\{-\tfrac{1}{2}\underline{x}^T(k)P^{-1}(k)\underline{x}(k)\}$$

where $P(k)$ is given by,

$$P(k+1) = \Phi(k+1,k)P(k)\Phi^T(k+1,k) + \Gamma(k+1)Q(k+1)\Gamma^T(k+1)$$

and it is assumed that $E[\underline{x}(0)] = \underline{0}$.

The following definition may then be made [57] :

D1: The state, $\underline{x}(k)$, of the system (II.4)-(II.5) is controllable if and only if $P(k)$ is non-singular for every $k>0$.

Alternatively, consider the difference at time k of the state vector $\underline{x}(k)$ and $\Phi(k,0)\underline{x}(0)$, given by,

$$\underline{d}(k) \triangleq \underline{x}(k) - \Phi(k,0)\underline{x}(0)$$

$$= \sum_{i=0}^{k-1} \Phi(k,i+1)\Gamma(i)\underline{w}(i)$$

Then, $E[\underline{d}(k)] = 0$; all k

and

$$\mathrm{cov}[\underline{d}(k)] = \sum_{i=0}^{k-1} \Phi(k,i+1)\Gamma(i)Q(i)\Gamma^T(i)\Phi^T(k,i+1)$$

$$\triangleq C(k,0)$$

If the norm of the matrix $C(k,0)$ remains bounded for all k, then $||\underline{d}(k)||$ will remain bounded for all k. In other words the effects of the random disturbances remain bounded.

The following alternative definition of stochastic system controllability may then be made *[40]*:

D2: The state $\underline{x}(k)$ of the system (II.4)-(II.5) is *completely controllable* if and only if

$$C(k,0) > 0, \text{ for some } k>0,$$

and *uniformly completely controllable* if there exist a positive integer N and positive constants α,β such that

$$0<\alpha I\leqslant C(k,k-N)\leqslant\beta I$$

The concept of observability refers to the information about the system state that can be gained from observing the system output. In stochastic systems observability may be defined as the existence condition of the system state estimates with certain specified asymptotic behaviours, where the class of the state estimates of $\underline{x}(k)$ is taken to be functions of y^k. The following definitions may then be made *[40]:*

D3: The state $\underline{x}(k)$ of the system (II.4)-(II.5) is said to be *observable* if and only if the covariance matrices associated with the conditional pdf of $\underline{x}(k)$ given y^k remain bounded in some sense.

Specifically, define the *information* or *observability matrix*

$$O(k,l) \triangleq \sum_{i=l}^{k} \Phi^T(i,k)H^T(k)R^{-1}(k)H(k)\Phi(i,k)$$

The observability matrix obeys the same equation as the state estimate error covariance matrix if the initial uncertainty is zero, i.e. $P(0)=0$. Then *[40]:*

D4: The state $\underline{x}(k)$ of the system (II.4)-(II.5) is *completely observable* if,

$$O(k,l) > 0 \text{ for some } k>0$$

It is *uniformly completely observable* if there exist a positive integer N and positive constants α, β such that,

$$0 < \alpha I \leqslant O(k,k-N) \leqslant \beta I \ .$$

II.3 Stochastic stability.

There exist many definitions of stability for deterministic systems. In general the intuitive idea of stability in a dynamical system is that for "small disturbances" from the equilibrium state at some time k, subsequent states should lie within a bounded region.

In stochastic systems, the stability of the Kalman filter is usually investigated. Let the following definitions be made *[40]*:

D5: The system (II.4)-(II.5) is *stable* if,

$$||\Phi(k,0)|| \leqslant c_1 \qquad \text{for all } k \geqslant 0$$

It is *asymptotically stable* if in addition,

$$\lim_{k\to\infty} ||\Phi(k,0)|| = 0$$

and *uniformly asymptotically stable* if

$$||\Phi(k,0)|| \leqslant c_2 \exp(-c_3 k) \quad \text{for all } k \geqslant 0, \text{ (the } c_i \text{ are positive constants).}$$

Consider the multivariable, time varying Kalman filter corresponding to (I.3)-(I.11). Then the filter state may be written in difference equation form as,

$$\underline{\hat{x}}(k/k) = [I-k(k)H(k)]\Phi(k,k-1)\underline{\hat{x}}(k-1/k-1) + K(k)\underline{y}(k)$$

and the following theorem holds *[40]*:

Theorem 1: If the system (II.4)-(II.5) is uniformly completelv controllable and uniformly completely observable and if $P(0)>0$ then the Kalman filter represents a uniformly asymptotically stable system.

As, a consequence,

$$||S(k,0)|| \leqslant c_2 \exp(-c_3 k)$$

where,

$$S(k,0) = \prod_{i=1}^{k} [I-K(i)H(i)]\Phi(i,i-1) \qquad \text{(II.6)}$$

II.4 System faults and fault monitoring.

In applying the discrete linear Kalman filter, described by equations (I.3)-(I.11), to a real system modelled by (II.1)-(II.2), the model parameters ϕ and η, noise variances q, r and initialisations $\hat{x}(0)$, $p(0)$ must be specified a-priori.

The performance of the Kalman filter will therefore depend on the accuracy of the model parameters, which are obtained by some identification procedure.

If the Kalman filter is operating with correctly identified parameters, then a sudden fault in the real system will introduce errors in the model parameters, which unless corrected, will produce degradation of filter performance. Such degradations may lead to an increase in the state estimate error variance $p(k/k)$ or a bias in the state estimate $\hat{x}(k/k)$ or both.

System faults may then be defined as follows:

D6: A *system fault* is any change in the assumed system parameters which causes a degradation in the performance of the state estimation procedure.

This definition of a fault implies some degradation of performance, in contrast with the term *failure*, which usually denotes complete inoperability of equipment or process.

The design objectives of a fault monitoring scheme consist of the required functions to be performed in the event of a fault and the performance requirements which are to be met. The following functions could be performed:

1. *Fault detection*, which simply consists of making a binary decision; either that a fault has occurred or not.

2. *Fault isolation*, which refers to the problem of determining the source of the fault where more than one parameter is subject to change.

3. *Fault estimation*, which involves the determination of the extent of the fault. This may be accomplished by estimating the time of the fault occurrence and the magnitude of the fault.

4. *System reorganization*, which entails reinitialisation of model and filter parameters according to decisions made within functions

1, 2, 3.

The extent to which functions 1-4 need be performed clearly depends upon the application. In some cases, it may be sufficient to generate an alarm, denoting faulty operation; further investigative action may then be taken by the human operator. In other systems, incorporating redundancy, fault isolation without estimation could be performed. Where hardware redundancy is not available, but faulty equipment may be capable of operating with reduced performance in a partial failure mode, then fault isolation with estimation would be necessary.

Intuitively, the tasks described can be associated with varying degrees of software system complexity- i.e. isolation requires more sophisticated data processing than an alarm, and estimation more than isolation. On the other hand, improved fault monitoring capabilities may result in a reduction in the requirement for hardware redundancy. Also in some applications, fault isolation and estimation may be delayed until after an alarm has been sounded. In such a sequential structure, the complexity of the monitoring process is increased only after a fault has been detected, thereby reducing the computational burden during normal, no-fault operation.

In addition, the fault monitoring scheme should possess the following properties:

1. The time taken to detect a fault, termed *detection delay time*, t_d, should be a low multiple of the system sampling time.

2. The *estimation delay time*, t_e, should also be small.

3. There must be a small *probability*, P_f, *of false alarms*.

4. There must be a high *probability*, P_d, *of correct detections*.

In the design of practical fault monitoring schemes, computational complexity is an important consideration. A scheme that has reasonable storage and time requirements is preferable. A design methodology which

incorporates options over a range of implementations is also useful, thus allowing a tradeoff between system complexity and performance. In addition it would be desirable to use a design approach which could take advantage of new computer capabilities and structures as they become available.

II.4.1 Fault modelling.

It is assumed that a fault can occur with equal probability at any time during the system operation. The size of the fault is also arbitrary but may be bounded above and/or below from physical considerations.

The general form of the fault model will be assumed additive, i.e. given a system parameter p and a fault modelled by $h(k,\theta,\nu)$ then the value of the parameter following a fault will be given by:

$$p_{new} = p_{old} + h(k,\theta,\nu)$$

where,

$$\nu \in [\nu_l, \nu_u]$$

is the size of the fault constrained below by ν_l and above by ν_u and

$$\theta \in [0, \infty)$$

is the time of fault occurrence which takes a finite integer value if a fault occurs and is infinite otherwise.

Faults may be classified into three types:

Type I: jump

Type II: step

Type III: ramp and higher order.

A *type I* fault may be modelled by the term

$$\nu\delta_{k,\theta}$$

where $\delta_{k,\theta}$ is the Kronecker delta defined by

$$\delta_{k,\theta} = 1;\ k=\theta$$

$$= 0;\ k\neq\theta$$

If a fault has not occurred θ is infinite, hence $\delta_{k,\infty} = 0$. In practice this model may be used for instantaneous faults of one time unit duration. This situation is illustrated in fig. 2.

A *type II* fault may be modelled by the term

$$\nu\sigma_{k,\theta}$$

where $\sigma_{k,\theta}$ is the unit step defined by,

$$\sigma_{k,\theta} = 1;\ k \geq \theta$$

$$= 0;\ k < \theta$$

If a fault has not occurred θ is infinite in which case $\sigma_{k,\infty} = 0$.

In practice this model may be used for faults of constant size which have a permanent effect on the system (fig. 3). Since estimation of a fault is carried out simultaneously with or following detection, it will be assumed that,

$$t_f > \max\{t_d, t_e\} = t_e$$

where t_f denotes the fault duration. Under this assumption consecutive steps could be monitored, since the fault monitoring process would have detected, estimated and subsequently reinitialised the filter parameters before the occurrence of a new fault.

A *type III* fault may be modelled by the term

$$h(\nu,k)\sigma_{k,\theta}$$

where $h(\nu,k)$ is a polynomial in k; ν may be used to represent faults of changing magnitude. A ramp could then be represented as

$$(a + k\nu)\sigma_{k,\theta} \qquad \text{(fig. 4)}.$$

Such models introduce further complexity to the fault monitoring scheme. However, there could be situations where ramps might be approximated by a series of steps. This approach will depend on the slope of the ramp, which should not be too steep for such an approximation to be

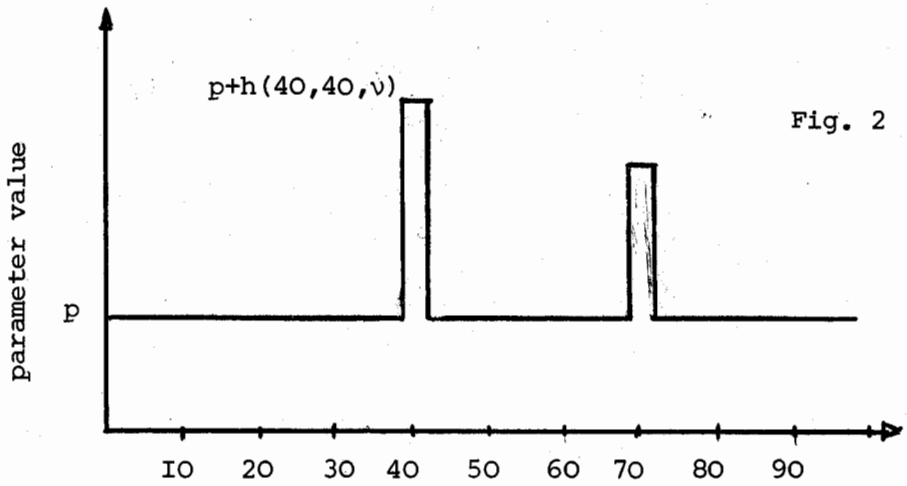

Fig. 2 Type I faults.

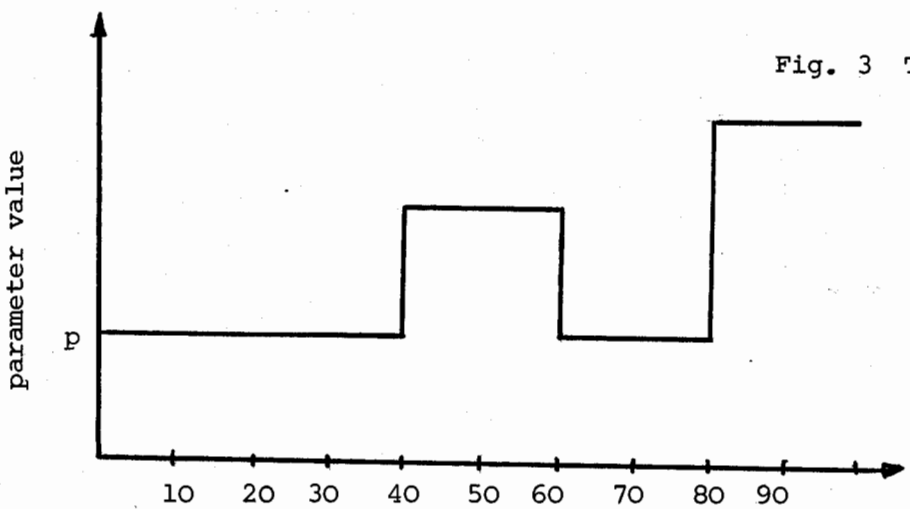

Fig. 3 Type II faults.

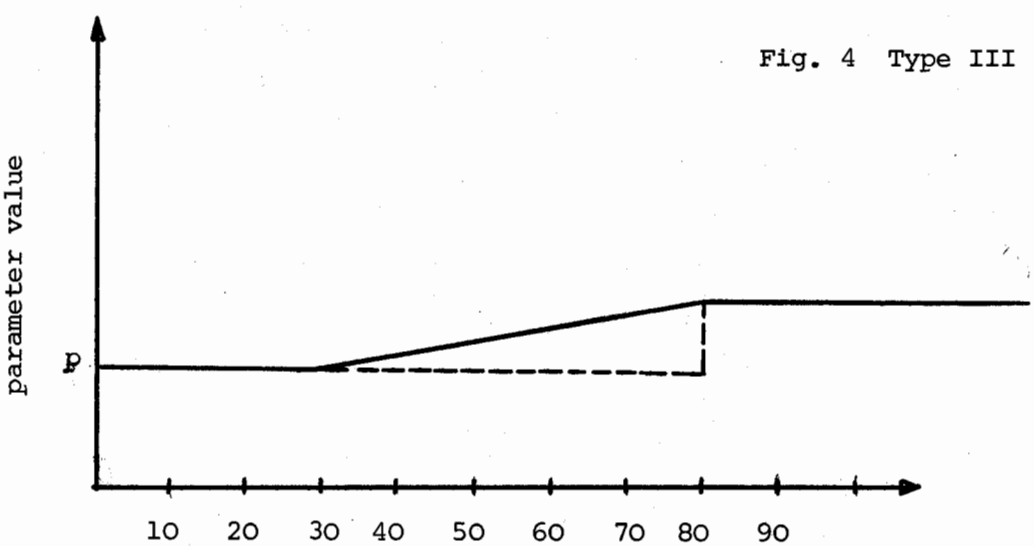

Fig. 4 Type III faults.

valid.

II.4.2 Modelling of systems subject to type II faults.

The system model, described by equations (I.1)-(I.2), can now be rewritten to include the possible effect of a fault. The fault models proposed are all of the additive type. They cover a range of situations which occur in practice, including: shifts in bias states in inertial systems *[32]*, *[33]*; leaks in jet thrusters in aerospace applications *[41]*; electrocardiogram arrythmias *[42]* and traffic control signalling *[9]*.

Faults of type I will have a temporary effect on the system performance since the Kalman filter will resettle at its pre-fault condition without the need for fault detection and estimation. Consider for example a type I fault in the value of ϕ. This may be modelled as,

$$x(k+1) = (\phi+\nu\delta_{k+1,\theta})x(k) + w(k)$$

$$= \phi x(k) + w(k) + \nu x(k)\delta_{k+1,\theta}$$

Assume $\theta=k+1$. At this time the Kalman filter estimate is $\hat{x}(k+1/k+1)$ with variance $p(k+1/k+1)$. The additional term $\nu x(k)$ introduced by the occurrence of the fault may be thought of as an error in specifying $\hat{x}(k+1/k+1)$. If the filter was started at time k+1, the effect of the initial error $\nu x(k)$ would diminish, under stability assumptions, as successive measurements are processed *[40]*.

The time taken for the Kalman filter to settle will in general be greater than t_e and therefore in applications where accuracy in state estimation is vital at every time, a fault monitoring scheme should be employed for faults of this type *[36]*.

Type II faults are more important since they will have a permanent effect on the performance of the filter and will thus have to be monitored. Once such a fault has been detected, isolated and estimated it will also be necessary to reinitialise the Kalman filter, which will then run with parameters which are modified following fault monitoring.

II.4.2.1 Type II faults in the parameters of the plant equation.

The following models are proposed in the case of faults in the plant equation:

a. *Step bias in plant state.*

$$x(k+1) = \phi x(k) + w(k) + \nu_x \sigma_{k+1,\theta} \qquad \text{(II.7)}$$

$$y(k) = \eta x(k) + v(k) \qquad \text{(II.8)}$$

Step changes in the mean of the plant noise sequence w(k) may also be modelled in this manner.

b. *Step change in* ϕ.

$$x(k+1) = (\phi + \Delta\phi\sigma_{k+1,\theta})x(k) + w(k) \qquad \text{(II.9)}$$

$$y(k) = \eta x(k) + v(k) \qquad \text{(II.10)}$$

c. *Additional plant noise.*

$$x(k+1) = \phi x(k) + w(k) + \zeta_x(k)\sigma_{k+1,\theta} \qquad \text{(II.11)}$$

$$y(k) = \eta x(k) + v(k) \qquad \text{(II.12)}$$

where $\zeta_x(k)$ is conveniently defined as a white gaussian random sequence, independent of x(0), w(i), v(i) for all i,k and of zero mean and unknown constant variance s_x.

Alternatively, an increase or decrease in the variance of the plant noise may be modelled as,

$$x(k+1) = \phi x(k) + (1+\nu_w \sigma_{k+1,\theta})w(k) \qquad \text{(II.13)}$$

$$y(k) = \eta x(k) + v(k) \qquad \text{(II.14)}$$

The new variance will be given by,

$$(1+\sigma_{k+1,\theta}\nu)^2 q$$

Hence if ν is negative (II.13) will model a decrease in the plant noise variance, whereas if n is positive an increase in the plant noise variance is modelled. This is particularly useful in identification applications of the described fault models where the initial guess for the noise variances may be optimistic or pessimistic. In fault detection applications however, where performance degradation is usually monitored the model given by equations (II.11)-(II.12) will be adequate.

II.4.2.2 Type II faults in the parameters of the measurement equation.

The following models are proposed in the case of faults in the parameters of the measurement equation:

d. *Step bias in the measurements.*

$$x(k+1) = \phi x(k) + w(k) \qquad \text{(II.15)}$$

$$y(k) = \eta x(k) + v(k) + \nu_y \sigma_{k,\theta} \qquad \text{(II.16)}$$

Step changes in the mean value of the measurement noise sequence v(k) may also be modelled in this manner.

e. *Step change in* η.

$$x(k+1) = \phi x(k) + w(k) \qquad \text{(II.17)}$$

$$y(k) = (\eta+\Delta\eta\sigma_{k,\theta})x(k) + v(k) \qquad \text{(II.18)}$$

f. *Additional measurement noise.*

$$x(k+1) = \phi x(k) + w(k) \qquad \text{(II.19)}$$

$$y(k) = \eta x(k) + v(k) + \zeta_y(k)\sigma_{k,\theta} \qquad \text{(II.20)}$$

where $\zeta_y(k)$ is conveniently defined as a gaussian sequence of zero mean and unknown constant variance s_y, independent of x(0), w(i), v(i) for all i,k.

Alternatively, an increase or decrease in the variance

of the measurement noise may be modelled as,

$$x(k+1) = \phi x(k) + w(k) \qquad \text{(II.21)}$$

$$y(k) = \eta x(k) + (1+\nu_v \sigma_{k+1,\theta})v(k) \qquad \text{(II.22)}$$

The same comments as for the case of additional plant noise apply here.

II.4.2.3 Single and multiple faults.

The models developed here may be used in situations where a fault may occur in only one parameter at any given instant. Such faults may be called *single* faults. The same approach can however be extended to the modelling of simultaneous occurrence of faults in more than one parameter. Such faults may be termed *multiple*.

Fault monitoring schemes for multiple faults will increase the computational complexity and cost due to the increase in the number of unknowns and the number of possible combinations of fault occurrences. If, for example three kinds of faults can occur in a given system, the number of possible faulty situations is seven. In general, the number of different fault possibilities is 2^n-1, where n is the number of distinct single faults that may occur.

Since in many applications it may be argued that the probability of occurrence of multiple faults is appreciably lower than for a single fault, the proposed monitoring schemes are based on the assumption of single faults.

CHAPTER III: *EFFECT OF FAULTS ON KALMAN FILTER RESIDUALS*

III.1 Effect of faults on the properties of the residual sequence.

Given the observability conditions, the true system is observable through the measurement sequence $\{y(k)\}$ only and equations (I.3)-(I.8) imply that knowing the measurement residual sequence $\{\gamma(k)\}$ is equivalent to knowing $\{y(k)\}$. It therefore follows that $\{\gamma(k)\}$ will contain information of faults provided that the faults are observable. Although $\{y(k)\}$ and $\{\gamma(k)\}$ both contain information of a fault, the use of $\{\gamma(k)\}$ for fault monitoring is fundamentally more attractive in a scheme based on statistical inference, since the residuals have been shown to be white , with zero mean while the measurements do not possess these properties *[46]*.

If no fault occurs, the residuals generated by the system measurements and the Kalman filter can be thought of as sample points from a normal probability distribution with zero mean and variance $c(k,k)$ given by (II.3). If a fault occurs the system output and therefore the residual sequence will no longer be represented by (I.2) and (I.6) respectively. The filter algorithm however, will still operate on the assumed values and as a consequence it will generate residuals which do not belong to the assumed probability distribution.

By obtaining the probability distribution of the residuals generated by the Kalman filter after each type of fault, fault detection and isolation can be performed by testing which of the possible probability distributions represent $\{\gamma(k)\}$. Estimation of the size of the fault can

then be performed by estimating a **parameter** of the appropriate probability distribution.

III.1.1 Effect of type-I faults in plant state.

It is shown in *[43]* that for a jump bias in the plant state modelled by,

$$x(k+1) = \phi x(k) + w(k) + \delta_{k+1,\theta}\nu$$

$$y(k) = \eta x(k) + v(k)$$

the state, measurement, state estimate and innovations sequences may be expressed as,

$$x(k) = x_o(k) + \phi^{k-\theta}\nu \qquad \text{(III.1)}$$

$$y(k) = y_o(k) + \eta\phi^{k-\theta}\nu \qquad \text{(III.2)}$$

$$\hat{x}(k/k) = \hat{x}_o(k/k) + f(k,\theta)\nu \qquad \text{(III.3)}$$

$$\gamma(k) = \gamma_o(k) + g(k,\theta)\nu \qquad \text{(III.4)}$$

where $x_o(k)$, $y_o(k)$, $\hat{x}_o(k/k)$ and $\gamma_o(k)$ represent the values of the corresponding variables that would be obtained if no fault occurs, and the additional terms exist if a fault occurs at time θ and are calculated from the recurrence relations,

$$g(k,\theta) = 0 \qquad \text{(III.5)}$$

$$f(k,\theta) = 0; \quad k<\theta \qquad \text{(III.6)}$$

$$g(\theta,\theta) = \eta \qquad \text{(III.7)}$$

$$f(\theta,\theta) = K(\theta)\eta; \quad k=\theta \qquad \text{(III.8)}$$

$$g(k,\theta) = \eta[\phi^{k-\theta} - \phi f(k-1,\theta)] \qquad \text{(III.9)}$$

$$f(k,\theta) = K(k)g(k,\theta) + \phi f(k-1,\theta); \quad k>\theta \qquad \text{(III.10)}$$

An important feature of these relations is the fact that the Kalman filter residuals, as indeed all parameters directly affected by the occurred fault, can be written as the sum of two terms, one of

which models effects solely due to θ and ν and the other represents all effects other than those due to θ and ν.

This result, obtained for the particular case of a jump bias in the plant state, may be generalised for any fault of the additive class.

III.1.2 Effect of additive faults.

The following theorem is a generalisation of the result of the previous section:

Theorem 2: The state, measurement, filter state estimate and innovations sequences for models represented by equations (II.1)-(II.2) and (I.3)-(I.11) which are subject to sudden faults modelled by any additive function, may be expressed as:

$$x(k) = x_o(k) + h_x(k,\theta,\Delta p) \qquad \text{(III.11)}$$

$$y(k) = y_o(k) + h_y(k,\theta,\Delta p) \qquad \text{(III.12)}$$

$$\hat{x}(k/k) = \hat{x}_o(k/k) + f(k,\theta,\Delta p) \qquad \text{(III.13)}$$

$$\gamma(k) = \gamma_o(k) + g(k,\theta,\Delta p) \qquad \text{(III.14)}$$

where,

$h_x(k,\theta,\Delta p)$ is the effect on state $x(k)$ of a fault of size Δp which occurred at time θ,

$h_y(k,\theta,\Delta p)$ is the corresponding effect on measurement $y(k)$,

$f(k,\theta,\Delta p)$ is the effect on the state estimate $\hat{x}(k/k)$ and

$g(k,\theta,\Delta p)$ is the effect on the residual $\gamma(k)$.

Further, the recursions on h_x, h_y, f and g are given by:

$$g(k,\theta,\Delta p) = h_y(k,\theta,\Delta p) - \eta\phi f(k-1,\theta,\Delta p) \qquad \text{(III.15)}$$

$$f(k,\theta,\Delta p) = K(k)g(k,\theta,\Delta p) + \phi f(k-1,\theta,\Delta p); \quad k\geq\theta \qquad \text{(III.16)}$$

$$g(k,\theta,\Delta p) = f(k,\theta,\Delta p) = 0; \quad k<\theta \qquad \text{(III.17)}$$

The proof is given in Appendix II.

The quantities h_x and h_y depend on the particular fault but in view of (I.2), if a fault occurs in a parameter of the plant equation

$$h_x(k,\theta,\Delta p) \neq 0 \quad \text{and} \qquad \text{(III.18)}$$

$$h_y(k,\theta,\Delta p) = \eta h_x(k,\theta,\Delta p); \quad k\geq\theta \qquad \text{(III.19)}$$

but if a fault occurs in a parameter of the measurement equation,

$$h_x(k,\theta,\Delta p) \equiv 0; \quad \text{all } k \qquad \text{(III.20)}$$

$$h_y(k,\theta,\Delta p) \neq 0; \quad k\geq\theta \qquad \text{(III.21)}$$

If a fault does not occur, since θ is infinite h_x and h_y are identically zero.

Equations (III.11)-(III.17) provide a model for the evolution of the $\{x(k)\}$, $\{y(k)\}$, $\{\hat{x}(k/k)\}$ and $\{\gamma(k)\}$. However, the state estimate and innovations sequences are still calculated by the Kalman filter using the real system measurements $y(k)$ from the equations,

$$\gamma(k) = y(k) - \eta\phi\hat{x}(k-1/k-1) \quad \text{and}$$

$$\hat{x}(k/k) = \phi\hat{x}(k-1/k-1) + K(k)\gamma(k)$$

The modelling however of the state estimate and innovations sequences by (III.13)-(III.14) is useful because it enables system performance to be analysed and checked under any fault condition that can be represented as an additional function, modelled by $h_y(k,\theta,\Delta p)$, in the measurements.

For type II faults, appropriate h_y functions can be calculated using the models developed in Sections II.4.2.1 and II.4.2.2. Alternatively, since the effects of faults are additive, type II faults

may be thought of as a series of successive type I faults. In this way, the effect of a type II fault of size Δp can be found by considering the total effect of successive type I faults of equal, but unknown, size Δp.

III.1.2.1 Step bias in plant state.

The jump bias equivalent for this kind of fault was presented in Section III.1.1.

The effect of consecutive jumps starting at time θ up to and including time k will therefore be:

$$g_a(k,\theta)\nu_x + g_a(k,\theta+1)\nu_x + \ldots + g_a(k,k)\nu_x$$

where, from (III.5)-(III.10),

$$g_a(i,j) = \eta[\phi^{i-j} - \phi f_a(i-1,j)] \qquad \text{(III.22)}$$

$$f_a(i,j) = K(i)g_a(i,j) + \phi f_a(i-1,j); \quad i \geq j \qquad \text{(III.23)}$$

$$g_a(i,j) = f_a(i,j) = 0; \quad i < j \qquad \text{(III.24)}$$

The residual sequence may then be written as:

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_a(k,i)\nu_x \qquad \text{(III.25)}$$

III.1.2.2 Step change in ϕ.

To calculate the appropriate g and f functions in the case of a change in the transition constant of the plant equation, consider equations (II.9)-(II.10) when $\theta=k+1$:

$$x(k+1) = (\phi+\Delta\phi)x(k) + w(k)$$

$$= x_o(k+1) + \Delta\phi x_o(k)$$

At time k+2, the following equations hold:

$$\begin{aligned} x(k+2) &= (\phi+\Delta\phi)x(k+1) + w(k+1) \\ &= (\phi+\Delta\phi)(x_o(k+1)+\Delta\phi x_o(k)) + w(k+1) \\ &= x_o(k+2) + \Delta\phi x_o(k+1) + (\phi+\Delta\phi)\Delta\phi x_o(k) \end{aligned}$$

It follows by inspection that the desired expression for the effect of a step change in ϕ is given by:

$$h_x(k,\theta,\Delta\phi) = \sum_{i=\theta}^{k} (\phi+\Delta\phi)^{k-i}\Delta\phi x_o(i-1) \qquad \text{(III.26)}$$

In this case, the effect of a step change cannot be written as the sum of effects of successive jump changes. Consider, for example a jump change at time k+1; then,

$$\begin{aligned} x(k+1) &= (\phi+\Delta\phi)x(k) + w(k) \\ &= x_o(k+1) + \Delta\phi x_o(k) \end{aligned}$$

The effect of this jump on the state at time k+2 is:

$$\begin{aligned} x(k+2) &= \phi x(k+1) + w(k+1) \\ &= \phi(x_o(k+1)+\Delta\phi x_o(k)) + w(k+1) \\ &= x_o(k+2) + \phi\Delta\phi x_o(k) \end{aligned} \qquad \text{(III.27)}$$

Similarly, if a jump occurs at time k+2, its effect is given by:

$$x(k+2) = x_o(k+2) + \Delta\phi x_o(k+1) \qquad \text{(III.28)}$$

Adding the two effects, given by (III.27) and (III.28), yields:

$$x(k+2) = x_o(k+2) + \Delta\phi x_o(k+1) + \phi\Delta\phi x_o(k) \qquad \text{(III.29)}$$

Using (III.26), the corresponding h_x function is:

$$h_x(k+2,k+1,\Delta\phi) = x_o(k+2) + \Delta\phi x_o(k+1) + (\phi+\Delta\phi)\Delta\phi x_o(k) \qquad \text{(III.30)}$$

Comparing (III.29) with (III.30) the assertion is seen.

However, the terms of the sum in (III.26) can be thought of as an effect of a jump fault. Therefore, define,

$$h_x^I(i,j,\Delta\phi) \triangleq (\phi+\Delta\phi)^{i-j}\Delta\phi x_o(j-1)$$

Equations (III.15)-(III.17) then become,

$$g_b'(i,j,\Delta\phi) = \eta h_x^I(i,j,\Delta\phi) - \eta\phi f_b'(i-1,j,\Delta\phi)$$
$$= \eta[(\phi+\Delta\phi)^{i-j}\Delta\phi x_o(j-1) - \phi f_b'(i-1,j,\Delta\phi)] \quad \text{(III.31)}$$

$$f_b'(i,j,\Delta\phi) = K(i)g_b'(i,j,\Delta\phi) + \phi f_b'(i-1,j,\Delta\phi); \quad i\geq j \quad \text{(III.32)}$$

$$g_b'(i,j,\Delta\phi) = f_b'(i,j,\Delta\phi) = 0; \quad i<j \quad \text{(III.33)}$$

Alternatively, since,

$$g_b'(j,j,\Delta\phi) = \eta\Delta\phi x_o(j-1) \text{ and}$$
$$f_b'(j,j,\Delta\phi) = K(j)\eta\Delta\phi x_o(j-1)$$

and the only term in (III.31)-(III.32) involving $\Delta\Delta\phi x_o(j-1)$ is h_x^I,

$$g_b'(i,j,\Delta\phi) = g_b(i,j,\Delta\phi)\Delta\phi x_o(j-1)$$
$$f_b'(i,j,\Delta\phi) = f_b(i,j,\Delta\phi)\Delta\phi x_o(j-1)$$

where g_b and f_b are recursively computed from:

$$g_b(i,j,\Delta\phi) = \eta[(\phi+\Delta\phi)^{i-j} - \phi f_b(i-1,j,\Delta\phi)] \quad \text{(III.34)}$$
$$f_b(i,j,\Delta\phi) = K(i)g_b(i,j,\Delta\phi) + \phi f_b(i-1,j,\Delta\phi); \quad i\geq j \quad \text{(III.35)}$$
$$g_b(i,j,\Delta\phi) = f_b(i,j,\Delta\phi) = 0; \quad i<j \quad \text{(III.36)}$$

It may be seen that in the case of a step change in ϕ, the unknown size of the fault is nonlinearly related to the additional terms g_b. As a consequence the form of equations (III.34)-(III.35)

cannot be written in the convenient form of equations (III.22)-(III.23).

Using equation (III.14), the filter residuals may be expressed as:

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_o(i-1) \qquad \text{(III.37)}$$

III.1.2.3 Additional plant noise.

In the case of additional plant noise the cumulative properties outlined in Section III.1.2 are valid. Consider equation (II.11) with $\theta=k+1$,

$$\begin{aligned} x(k+1) &= \phi x(k) + w(k) + \zeta_x(k) \\ &= x_o(k+1) + \zeta_x(k) \end{aligned}$$

and,

$$\begin{aligned} x(k+2) &= \phi x(k+1) + w(k+1) + \zeta_x(k+1) \\ &= x_o(k+2) + \zeta_x(k+1) + \phi\zeta_x(k) \end{aligned}$$

In general,

$$h_x(k,\theta,\zeta_x) = \sum_{i=\theta}^{k} \phi^{k-i}\zeta_x(i)$$

The terms in the above sum are just the sum of the effects of consecutive additional plant noise of jump type, since in this case

$$\begin{aligned} x(k+1) &= \phi x(k) + w(k) + \zeta_x(k) \\ &= x_o(k+1) + \zeta_x(k) \end{aligned}$$

and

$$\begin{aligned} x(k+2) &= \phi x(k+1) + w(k+1) \\ &= x_o(k+2) + \phi\zeta_x(k) \end{aligned}$$

In general, it may be seen that the effect is $\phi^{k-\theta}\zeta_x(\theta)$.

Hence, the residual sequence may be written as:

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_c(k,i)\zeta_x(i) \qquad \text{(III.38)}$$

where the g_c can be calculated iteratively using (III.15)-(III.16) with the appropriate h_y function of a jump fault, as:

$$g_c(i,j) = \eta[\phi^{i-j} - \phi f_c(i-1,j)] \qquad \text{(III.39)}$$

$$f_c(i,j) = K(i)g_c(i,j) + \phi f_c(i,j); \quad i \geq j \qquad \text{(III.40)}$$

$$g_c(i,j) = f_c(i,j) = 0; \quad i<j \qquad \text{(III.41)}$$

It may be seen that g_c obeys the same recurrence equations as g_a. This is to be expected as the sequence of additional plant noise can be thought of as a state bias of random magnitude.

III.1.2.4 Step bias in the measurements.

If a fault occurs in a parameter of the measurement equation simplification of the general results of Section III.1.2 is possible The cumulative property holds, since if a fault occurs, its effect is not propagated to subsequent states. For a step bias therefore, consider a series of successive jump biases. If a jump bias occurs at time k+1,

$$y(k+1) = \eta x(k+1) + v(k+1) + \nu_y$$
$$= y_o(k+1) + \nu_y$$

At subsequent times,

$$y(k+i) = y_o(k+i)$$

Hence, defining h_y^I as the effect of the corresponding type I fault,

$$h_y^I(i,j,\nu_y) = \nu_y; \quad i=j$$
$$= 0 ; \quad i \neq j$$

Hence, (III.15)-(III.17) become

$$g_d(i,j) = -\eta\phi f_d(i-1,j); \quad i>j \tag{III.42}$$

$$f_d(i,j) = K(i)g_d(i,j) + \phi f_d(i,j); \quad i\geq j \tag{III.43}$$

$$g_d(i,i) = 1 \tag{III.44}$$

$$g_d(i,j) = f_d(i,j) = 0; \quad i<j \tag{III.45}$$

and the residual sequence may be written,

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_d(k,i)\nu_y \tag{III.46}$$

Note that (III.42)-(III.45) should be strictly written with the factor ν_y included, as:

$$g_d(i,j) = 0 - \eta\phi f_d(i-1,j); \quad i>j \tag{III.47}$$

$$f_d(i,j) = K(i)g_d(i,j) + \phi f_d(i-1,j); \quad i\geq j \tag{III.48}$$

$$g_d(i,i) = \nu_y \tag{III.49}$$

$$g_d(i,j) = f_d(i,j) = 0; \quad i<j \tag{III.50}$$

However, it may be seen that (III.46) is an equivalent form to

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_d(k,i)$$

where the g_d are now defined by (III.47)-(III.50), and the form implied by (III.46) will be used in subsequent cases.

III.1.2.5 Step change in η.

In the case of a step change in η, the appropriate g functions can be calculated similarly to preceding cases by considering (III.18) with $\theta=k+1$:

$$y(k+1) = (\eta+\Delta\eta)x(k+1) + v(k+1)$$

$$= y_o(k+1) + \Delta\eta x_o(k+1)$$

Consequently,

$$h_y^I(i,j,\Delta\eta) = \Delta\eta\, x_o(j); \quad i=j$$
$$= 0; \quad i\neq j$$

Hence,

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_e(k,i)\Delta\eta x_o(i) \qquad \text{(III.51)}$$

where the g_e are defined, as the g_d, as:

$$g_e(i,j) = -\eta\phi f_e(i-1,j); \quad i>j \qquad \text{(III.52)}$$

$$f_e(i,j) = K(i)g_e(i,j) + \phi f_e(i-1,j); \quad i\geq j \qquad \text{(III.53)}$$

$$g_e(i,i) = 1 \qquad \text{(III.54)}$$

$$g_e(i,j) = f_e(i,j) = 0; \; i<j \qquad \text{(III.55)}$$

III.1.2.6 Additional measurement noise.

Considering (II.20) with $\theta=k+1$, gives:

$$y(k+1) = \eta x(k+1) + v(k+1) + \zeta_y(k+1)$$
$$= y_o(k+1) + \zeta_y(k+1)$$

hence,

$$h_y^I(i,j,\zeta_y) = \zeta_y(i); \quad i=j$$
$$= 0; \quad i\neq j$$

The residual sequence may then be written as,

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_f(k,i)\zeta_y(i) \qquad \text{(III.56)}$$

where g_f is similarly defined by (III.47)-(III.50) or (III.52)-(III.55).

III.1.2.7 Summary of results.

The results concerning the effect of the various kinds of type II faults on the form of the Kalman Filter residuals are summarised in Table 1.

As it has been shown in Sections III.1.2.1 - III.1.2.6, the innovations sequence can be written as the sum of two distinct terms. The first term is the value of the innovations if no fault occurs, while the second term models the effects of the fault only. Furthermore, the modelled effect of a type II fault, can be conveniently expressed as the sum of effects produced by successive corresponding type I faults (except in the case of a change in ϕ, where a non-corresponding type I fault has to be defined).

TABLE 1

Innovations modelling subject to type II faults.

	bias	additional noise	change in coefficient
plant	$\gamma(k)=\gamma_o(k)+\sum_{i=\theta}^{k} g_a(k,i)\nu_x$	$\gamma(k)=\gamma_o(k)+\sum_{i=\theta}^{k} g_c(k,i)\zeta_x(i)$	$\gamma(k)=\gamma_o(k)+\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_o(i-1)$
	g_a, g_c are both calculated by:		
	$g(i,j)=\eta[\phi^{i-j}-\phi f(i-1,j)]$		$g_b(i,j,\Delta\phi)=\eta[(\phi+\Delta\phi)^{i-j}-\phi f_b(i-1,j,\Delta\phi)]$
	$f(i,j)=K(i)g(i,j)+\phi f(i-1,j)$; $i\geq j$		$f_b(i,j,\Delta\phi)=k(i)g_b(i,j,\Delta\phi)+\phi f_b(i-1,j,\Delta\phi)$
	$g(i,j)=f(i,j)=0$; $i<j$		$g_b(i,j,\Delta\phi)=f_b(i,j,\Delta\phi)=0$; $i<j$
measurements	$\gamma(k)=\gamma_o(k)+\sum_{i=\theta}^{k} g_d(k,i)\nu_y$	$\gamma(k)=\gamma_o(k)+\sum_{i=\theta}^{k} g_f(k,i)\zeta_y(i)$	$\gamma(k)=\gamma_o(k)+\sum_{i=\theta}^{k} g_e(k,i)\Delta\eta x_o(i)$
	g_d, g_e, g_f are all calculated from:		
	$g(i,j)=-\eta\phi f(i-1,j)$; $i>j$		
	$f(i,j)=K(i)g(i,j)+\phi f(i-1,j)$; $i\geq j$		
	$g(i,i)=1$		
	$g(i,j)=f(i,j)=0$; $i<j$		

III.2 Effect of type II faults on the joint pdf of the innovations.

Having established the form of the innovations sequence under faulty conditions, their joint probability distribution will now be examined.

In normal operation the statistical properties of the residuals are given by (II.1)-(II.3). When a fault occurs, the residuals generated by the Kalman filter evolve according to (III.14). Therefore the joint pdf in the event of a fault occurrence can be calculated by considering (III.14) for every fault case.

For every possible fault, it is necessary to establish which of the statistical properties of the residuals in normal operation remain the same and which are subject to change.

Since the linear structure of the Kalman filter equations and state and measurement models is not changed in the presence of an additive type fault, the residuals remain a linear combination of the gaussian measurement sequence $\{y(k)\}$ and are therefore also gaussian. This result means that the joint pdf of the innovations will be completely characterised by its first and second moments. The effect of the faults on the whiteness property must be examined as well. In normal operation the whiteness property enables the joint pdf of the residuals to be written as the product of the individual pdf's of each residual. If this property does not hold, an orthogonalisation procedure may be employed.

To ease notational complexity the following definitions are made:

$$\underline{\gamma}^{j,k} \triangleq [\gamma(j), \gamma(j+1), \ldots, \gamma(k)]^T \tag{III.57}$$

$$\bar{\underline{\gamma}}^{j,k} \triangleq E\{[\gamma(j), \gamma(j+1), \ldots, \gamma(k)]^T\} \tag{III.58}$$

$$= [E[\gamma(j)], E[\gamma(j+1)], \ldots, E[\gamma(k)]]^T$$

$$C^{j,k} \triangleq \text{cov}[(\underline{\gamma}^{j,k}), \underline{\gamma}^{j,k}] \tag{III.59}$$

$$= E[(\underline{\gamma}^{j,k} - \bar{\underline{\gamma}}^{j,k})(\underline{\gamma}^{j,k} - \bar{\underline{\gamma}}^{j,k})^T]$$

Using these definitions the pdf of the gaussian vector $\underline{\gamma}^{j,k}$ is,

$$p(\underline{\gamma}^{j,k}) = \frac{1}{(2\pi)^{\frac{1}{2}n} |C^{j,k}|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2} (\underline{\gamma}^{j,k} - \bar{\underline{\gamma}}^{j,k})^T (C^{j,k})^{-1} (\underline{\gamma}^{j,k} - \bar{\underline{\gamma}}^{j,k})\} \tag{III.60}$$

where n=k-j+1 is the dimension of the residual vector.

If a fault has not occurred (III.60) becomes:

$$p(\underline{\gamma}^{j,k}) = \prod_{m=j}^{k} \frac{1}{(2\pi c(m,m))^{\frac{1}{2}}} \exp\{-\tfrac{1}{2} \frac{\gamma^2(m)}{c(m,m)}\} \tag{III.61}$$

$$\triangleq \pi(j,k) \tag{III.62}$$

III.2.1 Joint pdf in the event of step bias in plant state.

The residual sequence in the event of a step bias in the state is given by (III.25) as:

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_a(k,i) \nu_x$$

Hence,

$$E[\gamma(k)] = E[\gamma_o(k) + \sum_{i=\theta}^{k} g_a(k,i) \nu_x]$$

and since the residuals in normal operation have zero mean and the second term in the expectation is non-random,

$$E[\gamma(k)] = \sum_{i=\theta}^{k} g_a(k,i) \nu_x$$

Therefore the residual mean vector is:

$$\underline{\bar{\gamma}}_a^{j,k} = [0,\ 0,\ldots,\ g_a(\theta,\theta)\nu_x,\ldots,\ \sum_{i=\theta}^{k} g_a(k,i)\nu_x]^T \quad \text{(III.63)}$$

In the steady state following a fault, i.e. when $k>>\theta$, it is shown in Appendix I.1(a) that,

$$\bar{\gamma}(k) = \frac{\eta\nu_x}{1-(1-K\eta)\phi}$$

$$= \frac{\eta\nu_x}{1-s} \quad \text{(III.64)}$$

where $$s \triangleq (1-K\eta)\phi \quad \text{(III.65)}$$

and K is the steady state value of the Kalman filter gain given by (I.5).

The residual covariance matrix can be calculated considering,

$$E[(\gamma(k)-\bar{\gamma}(k))(\gamma(m)-\bar{\gamma}(m))]$$

But, $$\gamma(k)-\bar{\gamma}(k) = \gamma_0(k) + \sum_{i=\theta}^{k} g_a(k,i)\nu_x - \sum_{i=\theta}^{k} g_a(k,i)\nu_x$$

$$= \gamma_0(k)$$

Hence,

$$\text{cov}[\gamma(k),\gamma(m)] = 0; \quad k\neq m$$

$$= c(k,k); \quad k=m$$

This result implies that a step bias in the plant state does not change the correlation properties of the innovations sequence. The joint pdf of the residual sequence may then be written as:

$$p(\underline{\gamma}^{j,k}) = \pi(j,\theta-1)\prod_{i=\theta}^{k}\frac{1}{(2\pi c(i,i))^{\frac{1}{2}}}\exp\left\{-\tfrac{1}{2}\frac{\{\gamma(i)-\sum_{m=\theta}^{i} g_a(i,m)\nu_x\}^2}{c(i,i)}\right\}$$

(III.66)

where $\pi(j,\theta-1)$ is defined by (III.62).

III.2.2 Joint pdf of residuals in the event of a step change in ϕ.

The residual sequence in the event of a step change in ϕ, is given by (III.37) as:

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_o(i-1)$$

where $g_b(k,i,\Delta\phi)$ is defined by (III.34)-(III.36). Hence, in this case,

$$E[\gamma(k)] = E[\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_o(i-1)]$$

$$= \sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi E[x_o(i-1)]$$

Using (I.1),

$$E[x_o(i-1)] = \phi E[x_o(i-2)] + E[w(i-2)]$$

$$= \phi E[x_o(i-2)]$$

$$= \phi^{i-1}\ \bar{x}(0)$$

Under system stability assumptions $|\phi|<1$, hence in system steady state, $E[x_o(i)]\neq 0$; all $i>>0$. This result implies that if g_b remains bounded for all k, the mean value of the residual sequence in the event of a change in ϕ is zero if the fault occurs when the system has reached steady state. It is also shown in Appendix I.1(b) that g_b will remain bounded if $|\phi+\Delta\phi|<1$, that is if the change does not destabilize the system. Under these conditions, the residual covariance is,

$$\text{cov}[\gamma(k),\gamma(m)] = E[\gamma(k)\gamma(m)]$$

$$= E[(\gamma_o(k)+\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_o(i-1)(\gamma_o(m)+\sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\Delta\Phi x_o(j-1))]$$

$$\text{or } \operatorname{cov}[\gamma(k)\gamma(m)] = E[\gamma_o(k)\gamma_o(m)] + \sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\Delta\phi E[\gamma_o(k)x_o(j-1)] + \sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi E[x_o(i-1)\gamma_o(m)] + E\left[\left\{\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_o(i-1)\right\}\left\{\sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\Delta\phi x_o(j-1)\right\}\right] \quad \text{(III.67)}$$

The above terms, the summations in particular, may be calculated using the fact that a stochastic process x(k) obeying (I.1) has the following properties *[48]*:

$$\operatorname{cov}[x(k),x(m)] = \phi^{m-k} \operatorname{var}[x(k)]; \quad k<m$$

and,

$$\operatorname{var}[x(k)] = \phi^{2k}\operatorname{var}[x(0)] + \sum_{i=0}^{k-1}\phi^{2i}q$$

$$= \phi^{2k}p(0) + \sum_{i=0}^{k-1}\phi^{2i}q \quad \text{(III.68)}$$

The first term in (III.68) vanishes under stability conditions in steady state, while the second is a geometric sum whose value is:

$$\sum_{i=0}^{k-1}\phi^{2i}q = q\,\frac{1-\phi^{k}}{1-\phi^{2}} = \frac{q}{1-\phi^{2}}, \text{ when } k>>0.$$

hence, in system steady state,

$$\operatorname{var}[x(k)] = \frac{q}{1-\phi^{2}} \quad \text{(III.69)}$$

and,

$$\operatorname{cov}[x(k),x(m)] = \frac{\phi^{m-k}q}{1-\phi^{2}} \quad \text{(III.70)}$$

Also, since the $\gamma_o(i)$ are independent, $\gamma_o(k)$ is independent of $x_o(i)$ for all $i>k$. This fact makes the third sum in (III.67) vanish.

The last sum is:

$$g_b(k,\theta,\Delta\phi)\Delta\phi \sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\Delta\phi E[x_o(\theta-1)x_o(j-1)] +$$

$$+ g_b(k,\theta+1,\Delta\phi)\Delta\phi \sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\Delta\phi E[x_o(\theta)x_o(j-1)] +$$

$$+ \ldots. +$$

$$+ g_b(k,k,\Delta\phi)\Delta\phi \sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\Delta\phi E[x_o(k-1)x_o(j-1)] \qquad \text{(III.71)}$$

The expectations can be calculated using (III.68):

$$E[x_o(i)x_o(j)] = \phi^{|i-j|}\left(\phi^{2\lambda}p(o) + q\sum_{n=0}^{\lambda-1}\phi^{2n}\right) \qquad \text{(III.72)}$$

where $\lambda=\min\{i,j\}$.

Substituting (III.72) in (III.71) yields,

$$(\Delta\phi)^2 g_b(k,\theta,\Delta\phi)\sum_{j=\theta}^{m}\{g_b(m,j,\Delta\phi)\phi^{|j-\theta|}(\phi^{2\lambda_\theta}p(0)+q\sum_{n=0}^{\lambda_\theta-1}\phi^{2n})\} +$$

$$+\ldots+$$

$$+ (\Delta\phi)^2 g_b(k,k,\Delta\phi)\sum_{j=\theta}^{m}\{g_b(m,j,\Delta\phi)\phi^{|j-k|}(\phi^{2\lambda_k}p(0)+q\sum_{n=0}^{\lambda_k-1}\phi^{2n})\}$$

$$= (\Delta\phi)^2\sum_{i=\theta}^{k}\{g_b(k,i,\Delta\phi)\{\sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\phi^{|i-j|}(\phi^{2\lambda_i}p(0)+q\sum_{n=0}^{\lambda_i-1}\phi^{2n})\}\} \qquad \text{(III.73)}$$

where $\lambda_i=\min\{i,j\}$.

The second sum in (III.67) is

$$\sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\Delta\phi E[\gamma_o(k)x_o(j-1)] \qquad \text{(III.74)}$$

Since $\gamma_o(k)$ is independent of $x_o(i)$, $i=0,\ldots,k-1$, (III.67) becomes,

$$\sum_{j=k+1}^{m} g_b(m,j,\Delta\phi)\Delta\phi E[\gamma_o(k)x_o(j-1)]$$

The expectation is:

$$E[\gamma_o(k)x_o(j-1)] = E[\gamma_o(k)\{\phi^{j-1-k}x_o(k) + \sum_{n=k}^{i-1} a_i w(i)\}]$$

where the a_i are real constants. Since the w(i) are independent,

$$\begin{aligned} E[\gamma_o(k)x_o(j-1)] &= \phi^{j-1-k}E[\gamma_o(k)x_o(k)] \\ &= \phi^{j-1-k}E[\{\eta x_o(k)+v(k)-\eta\hat{x}_o(k/k-1)\}x_o(k)] \\ &= \phi^{j-1-k}\eta p(k/k) \end{aligned}$$

Finally (III.74) becomes,

$$\sum_{j=k+1}^{m} g_b(m,j,\Delta\phi)\Delta\phi\phi^{j-1-k}\eta p(k/k) \qquad \text{(III.75)}$$

Adding (III.75), (III.73) and c(k,m) gives the desired expression for the correlation between successive residuals in the event of a change in the plant transition coefficient:

$$\begin{aligned} E[\gamma(k)\gamma(m)] = c(k,m) &+ \\ &+ \sum_{j=k+1}^{m} g_b(m,j,\Delta\phi)\Delta\phi\phi^{j-1-k}\eta p(k/k) + \\ &+ (\Delta\phi)^2 \sum_{i=\theta}^{k} \{g_b(k,i,\Delta\phi)\{\sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\phi^{|i-j|}(\phi^{2\lambda_i}p(0)+q\sum_{n=0}^{\lambda_i-1}\phi^{2n})\}\} \end{aligned}$$

(III.76)

III.2.3 Joint pdf of residuals in the event of additional plant noise.

The residual sequence in the event of additional plant noise is given by (III.38) as,

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_c(k,i)\zeta_x(i)$$

The expected value is zero, since both $\gamma_o(k)$ and $\zeta_x(i)$ are of zero mean. The covariance is given by:

$$\text{cov}[\gamma_o(k)\gamma_o(m)] = E[\{\gamma_o(k) + \sum_{i=\theta}^{k} g_c(k,i)\zeta_x(i)\}\{\gamma_o(m) + \sum_{j=\theta}^{m} g_c(m,j)\gamma_x(j)\}]$$

Since,

$E[\zeta_x(j)v(i)] = 0$; all i,j and $E[\zeta_x(i)\zeta_x(j)] = 0$; all $i \neq j$,

the $\zeta_x(i)$ are independent of the $\gamma_o(i)$. Hence,

$$\text{cov}[\gamma_o(k),\gamma_o(m)] = c(k,m) + \sum_{i=\theta}^{\lambda} g_c(k,i)g_c(m,i)s_x$$

where $\lambda=\min\{k,m\}$.

The residual covariance matrix is then given by:

$$C_c^{j,k} = \left[\begin{array}{c|c} C^{j,\theta-1} & 0 \\ \hline 0 & C_c^{\theta,k} \end{array}\right]$$

where,

$$C_c^{\theta,k} = \begin{bmatrix} c(\theta,\theta)+c_c'(\theta,\theta) & c_c'(\theta+1,\theta) & \cdots & c_c'(\theta,k) \\ c_c'(\theta+1,\theta) & c(\theta+1,\theta+1)+c_c'(\theta+1,\theta+1) & \cdots & c_c'(\theta+1,k) \\ \vdots & & \ddots & \vdots \\ c_c'(\theta,k) & \cdots & & c(k,k)+c_c'(k,k) \end{bmatrix}$$

and $$c_c'(i,j) = \sum_{m=\theta}^{\lambda} g_c(i,m)g_c(j,m)s_x \qquad \text{(III.77)}$$

It can be seen from the form of (III.77) that the residual sequence following an increase in the plant noise is not stationary as well as not white, since in general,

$$c_c(i,j) \neq c_c(i+m,j+m)$$

However, in steady state following a fault, it is shown in Appendix I.1(c) that

$$c_c'(i,j) = c_c'(i+m,j+m) = c_c'(i-j)$$
$$= s^{i-j} \frac{\eta^2}{1-s^2} s_x \qquad \text{(III.78)}$$

where s is defined by (III.65). The form of (III.78) implies that under filter stability assumptions,

$$\lim_{(i-j)\to\infty} c_c'(i-j) = 0$$

i.e. the correlation between residuals following an increase in plant noise decreases exponentially with the distance between them. In steady state following the fault, the covariance matrix may therefore be written:

$$C_c^{j,k} = \begin{bmatrix} c(j,j)+c_c(0) & c_c(1) & \cdots & c_c(k) \\ c_c(1) & c(j+1,j+1)+c_c(0) & \cdots & c_c(k-1) \\ \vdots & & \ddots & \vdots \\ c_c(k) & c_c(k-1) & \cdots & c(k,k)+c_c(0) \end{bmatrix}$$

The joint pdf in this case is:

$$p(\underline{\gamma}^{j,k}) = \pi(j,\theta-1) \frac{1}{2\pi^{\frac{1}{2}(k-\theta+1)} |C_c^{\theta,k}|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}[\underline{\gamma}^{\theta,k}]^T [C_c^{\theta,k}]^{-1} \underline{\gamma}^{\theta,k}\} \qquad \text{(III.79)}$$

III.2.4 Joint pdf of residuals in the event of step bias in the measurements.

The residual sequence in the event of a step bias in the measurements is given by,

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_d(k,i) \nu_y$$

The expected value and covariance of the innovations sequence are found in the same way as the case of a step bias in the state. Thus,

$$E[\gamma(k)] = \sum_{i=\theta}^{k} g_d(k,i)\nu_y$$

and the mean value vector is,

$$\underline{\gamma}_d^{j,k} = [0,\ 0,\ \ldots,\ g_d(\theta,\theta)\nu_x,\ \ldots,\ \sum_{i=\theta}^{k} g_d(k,i)\nu_x]^T \quad \text{(III.80)}$$

For $k>>\theta$, it is shown in Appendix I.1(d) that

$$E[\gamma(k)] = \left(1 + \frac{\eta\phi k}{s-1}\right)\nu_y \quad \text{(III.81)}$$

Since the step bias is non-random the correlation properties do not change,

$$\text{cov}[\gamma(k),\gamma(m)] = E[\gamma_o(k)\gamma_o(m)] = c(k,m)$$

The results obtained in the case of a step bias in the measurements, which are qualitatively the same as those obtained in the case of a step bias in the state, imply that the joint pdf can be written as a product of independent random variables as:

$$p(\underline{\gamma}^{j,k}) = \pi(j,\theta-1)\prod_{i=\theta}^{k} \frac{1}{(2\pi c(i,i))^{\frac{1}{2}}} \exp\left\{-\tfrac{1}{2}\,\frac{[\gamma(i) - \sum_{m=\theta}^{i} g_d(i,m)\nu_x]^2}{c(i,i)}\right\} \quad \text{(III.82)}$$

III.2.5 Joint pdf of residuals in the event of a step change in η.

The residual sequence in the event of a step change in the measurement coefficient is given by (III.51) as:

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_e(k,i)\Delta\eta x_o(i)$$

where the g_e are defined by (III.52)-(III.55).

Since the $x_o(i)$ are random and E is linear,

$$E[\gamma(k)] = \sum_{i=\theta}^{k} g_e(k,i)\Delta\eta E[x_o(i)]$$

It was shown in Section III.2.2 that if the system has reached steady state when the fault occurs, $E[x_o(i)]=0$, $i>>0$. Hence, under conditions of steady state,

$$E[\gamma(k)] = 0$$

The covariance is calculated by,

$$E[\gamma(k)\gamma(m)] = E[\{\gamma_o(k)+\sum_{i=\theta}^{k} g_e(k,i)\Delta\eta x_o(i)\}\{\gamma_o(m)+\sum_{j=\theta}^{m} g_e(m,j)\Delta\eta x_o(j)\}]$$

$$= E[\gamma_o(k)\gamma_o(m)] +$$

$$+ \sum_{j=\theta}^{m} g_e(m,j)\Delta\eta E[\gamma_o(k)x_o(j)] +$$

$$+ \sum_{i=\theta}^{k} g_e(k,i)\Delta\eta E[\gamma_o(m)x_o(i)] +$$

$$+ E[\{\sum_{i=\theta}^{k} g_e(k,i)\Delta\eta x_o(i)\}\{\sum_{j=\theta}^{m} g_e(m,j)\Delta\eta x_o(j)\}] \quad \text{(III.83)}$$

The individual sums are of the same form as the sums involved in the covariance function of the residuals following a step change in ϕ. Thus, using (III.73), (III.75), the last sum in (III.83) is,

$$(\Delta\eta)^2 \sum_{i=\theta}^{k}\{g_e(k,i)\sum_{j=\theta}^{m}\{g_e(m,j)\phi^{|i-j|}(\phi^{2\lambda_i}p(0)+q\sum_{n=0}^{\lambda_i}\phi^{2n})\}\} \quad \text{(III.84)}$$

where $\lambda_i=\min\{i,j\}$.

The third sum vanishes, while the second is,

$$\sum_{j=k}^{m} g_e(k,i)\Delta\eta\phi^{j-1-k}\eta p(k/k) \quad \text{(III.85)}$$

Adding (III.84), (III.85) yields the desired expression for the residual covariance following a step change in the coefficient of the measurement equation:

$$E[\gamma(k)\gamma(m)] = c(k,m) + \\ + \sum_{j=k}^{m} g_e(m,j)\Delta\eta\phi^{j-1-k}\eta p(k/k) + \\ + (\Delta\eta)^2 \sum_{i=\theta}^{k}\{g_e(k,i)\sum_{j=\theta}^{m}\{g_e(m,j)\phi^{|i-j|}(\phi^{2\lambda_i}p(0)+q\sum_{n=0}^{\lambda_i}\phi^{2n})\}\} \qquad \text{(III.86)}$$

In this case therefore, as well as in the case of a change in ϕ, if the fault occurs when the system has reached steady state, the residuals retain their zero mean property but become correlated. It may be seen that since the g_b and g_e are bounded and ϕ is stable, the correlation given by both (III.76) and (III.86) reaches a steady state value.

III.2.6 Joint pdf of residuals in the event of additional measurement noise.

The residual sequence in the event of additional measurement noise is given by (III.56) as,

$$\gamma(k) = \gamma_o(k) + \sum_{i=\theta}^{k} g_f(k,i)\zeta_y(i)$$

Since the $\zeta_y(i)$ have zero mean,

$$E[\gamma(k)] = 0$$

The covariance is given by:

$$\text{cov}[\gamma(k)\gamma(m)] = E[\{\gamma_o(k)+\sum_{i=\theta}^{k} g_f(k,i)\zeta_y(i)\}\{\gamma_o(m)+\sum_{j=\theta}^{m} g_f(m,j)\zeta_y(j)\}]$$

Since $\zeta_y(i)$ is independent of $v(i)$, all i, $\zeta_y(i)$ independent of $\zeta_y(j)$, all $i \neq j$, it follows that $\zeta_y(i)$ is independent of $\gamma(i)$. Hence,

$$E[\gamma(k)\gamma(m)] = c(k,m) + \sum_{i=\theta}^{\lambda} g_f(k,i) g_f(m,i) s_y \qquad \text{(III.87)}$$

where $\lambda = \min\{k,m\}$.

This result implies that an increase in the measurement noise has the same qualitative effect on the joint pdf of the innovations sequence as the increase in plant noise, i.e. the residuals remain zero mean but become correlated and non-stationary.

In steady state following a fault however, it is shown in Appendix I.1(e) that,

$$\text{cov}[\gamma(k),\gamma(m)] = -\eta\phi K s^{m-k-1} \left\{ \frac{\eta\phi K s^{-1}}{1 - s^{-2}} + 1 \right\} s_y; \quad k \neq m \qquad \text{(III.88)}$$

$$= c + \left\{ 1 - \frac{(\eta\phi K)^2}{s^2 - 1} \right\} s_y; \quad k = m \qquad \text{(III.89)}$$

where c is the steady state value of $c(k,k)$ and s is defined by (III.65). Also, as $m-k \to \infty$, the correlation tends to zero. This result shows that in steady state following a fault, the correlation between the residuals decreases exponentially.

The joint pdf of the residual sequence will consist of a pre-fault residual string and a post fault correlated residual sequence,

$$p(\underline{\gamma}^{j,k}) = \pi(j,\theta-1) \frac{1}{2\pi^{\frac{1}{2}(k-\theta+1)} |C_f^{\theta,k}|^{\frac{1}{2}}} \exp\left\{ -\tfrac{1}{2} [\underline{\gamma}^{\theta,k}]^T [C_f^{\theta,k}]^{-1} \underline{\gamma}^{\theta,k} \right\} \qquad \text{(III.90)}$$

where the matrix $C_f^{\theta,k}$ is defined by:

$$C_f^{\theta,k} = \begin{bmatrix} c(\theta,\theta)+c_f(\theta,\theta) & c_f(\theta+1,k) & \cdots & c_f(\theta,k) \\ c_f(\theta+1,k) & c(\theta+1,\theta+1)+c_f(\theta+1,\theta+1) & \cdots & c_f(\theta+1,k) \\ \vdots & & \ddots & \vdots \\ c_f(\theta,k) & c_f(\theta+1,k) & \cdots & c(k,k)+c_f(k,k) \end{bmatrix} \quad \text{(III.91)}$$

and,

$$c_f(i,j) = \sum_{m=\theta}^{\lambda} g_f(i,m)\, g_f(j,m)\, s_y$$

where $\lambda=\min\{i,j\}$ and the g_f are defined by (III.56).

III.2.7 Summary of results and comments.

The effects of the type II faults are summarised in Tables 2 and 3. As it can be seen from Table 2, if the faults occur in steady state, then they may be classified into two disjoint classes, as follows:

C_1: {faults with effect of nonzero mean of residuals}

C_2: {faults with effect of correlated residuals}

or, equivalently,

C_1: {faults a, d} (III.92)

C_2: {faults b, c, e, f} (III.93)

The no-fault class,

C_0: {no fault} (III.94)

may also be added, so that the three classes fully characterise any probable condition of the system.

The results concerning the stationarity property of the residuals, in steady state following a fault are quite important. This property,

TABLE 2

Effect of faults on innovations in steady state conditions of system and filter.

	zero mean	independence	stationarity
a. State bias	no	yes	yes
b. Change in ϕ	yes	no	
c. Additional plant noise	yes	no	no(yes)
d. Measurement bias	no	yes	yes
e. Change in η	yes	no	
f. Additional measurement noise	yes	no	no(yes)
No fault	yes	yes	yes

In the case of additional noise, either in the state or the measurements, the entries in parentheses denote the steady state

TABLE 3

Steady state values of mean and correlation of residuals following a fault.

	mean	variance	correlation of $\gamma(i)\gamma(j)$
a. State bias	$\frac{\eta\nu_x}{1-s}$		
d. Measurement bias	$\{1+\frac{\eta\phi K}{s-1}\}\nu_y$		
c. Additional plant noise		$\frac{\eta^2 s_x}{1-s^2}+c$	$\frac{s^{i-j}\eta^2}{1-s^2}s_x$
f. Additional measurement noise		$\{1-\frac{(\eta\phi K)^2}{s^2-1}\}s_y+c$	$-\eta\phi K s^{i-j-1}\{\frac{\eta\phi K s^{-1}}{1-s^{-2}}+1\}s_y$

K and c represent the steady state values of the Kalman filter gain and residual variance respectively, while $s=(1-K\eta)\phi$.

together with the fact that the correlation decreases exponentially, ensures that time averages are meaningful *[49]*. Thus, even under faulty conditions,the sequence of residual values can be considered to be ensemble values of the corresponding distributions and hence on-line fault monitoring is possible.

A fourth class could also be included, covering cases outside the main assumptions of the problem. This would include situations where a fault in the transition coefficients occurs when the system is in the transient state or situations in which $|\phi+\Delta\phi|>1$, i.e. the change in ϕ destabilises the system. The common feature of the effect on the innovations sequence of faults of this class, is the introduction of bias as well as correlation. Therefore, C_3 may be defined as:

C_3: {faults with effect of nonzero mean and correlation}

or equivalently,

C_3: {(b or e in transient state) or (destabilising b)}

III.2.8 A multivariable extension.

The extension of the results obtained in Section III.1.2 to the multivariable case is straightforward. Time-varying systems can also be treated. The case of additional measurement noise will be developed as an example.

The multivariable time-varying version of equations (III.19)-(III.20) is the set of equations described by,

$$\underline{x}(k+1) = \Phi(k+1,k)\underline{x}(k) + \underline{w}(k) \qquad \text{(III.95)}$$

$$\underline{y}(k) = H(k)\underline{x}(k) + \underline{v}(k) + \sigma_{k,\theta}\underline{\zeta}_y(k) \qquad \text{(III.96)}$$

where Φ and H are parameter matrices of appropriate dimensions, and $\underline{w}(k)$, $\underline{v}(k)$, $\underline{\zeta}_y(k)$ are mutually independent random gaussian vectors, with zero mean and covariance matrices $Q(k)$, $R(k)$, $S_y(k)$ respectively. Each sequence is assumed white and independent of the initial state $\underline{x}(0)$. Under these assumptions, it is shown in Appendix I.2 that, corresponding to equations (III.56) and (III.52)-(III.55), the following equations hold in the multivariable, time-varying case:

$$\underline{\gamma}(k) = \underline{\gamma}_o(k) + \sum_{i=\theta}^{k} G(k,i)\underline{\zeta}_y(i)$$

where the matrices G(k,i) are calculated from:

$$G(i,j) = -H(i)\Phi(i,i-1)F(i-1,j); \quad i>j$$

$$F(i,j) = \Phi(i,i-1)F(i-1,j) + K(i)G(i,j); \quad i\geq j$$

$$G(i,i) = I$$

$$G(i,j) = F(i,j) = 0; \quad i<j$$

where K(i) is the Kalman filter gain matrix.

Therefore, since the $\underline{\zeta}_y(i)$ have zero mean,

$$E[\underline{\gamma}(k)] = \underline{0}$$

and,

$$\text{cov}[\underline{\gamma}(k),\underline{\gamma}(m)] = E[\{\underline{\gamma}_0(k)+\sum_{i=\theta}^{k} G(k,i)\underline{\zeta}_y(i)\}\{\underline{\gamma}_0(m)+\sum_{j=\theta}^{m} G(m,j)\underline{\zeta}_y(j)\}^T]$$

$$= E[\underline{\gamma}_0(k)\underline{\gamma}_0^T(m)] + \sum_{i=\theta}^{\lambda} G(k,i)S_y(i)G^T(m,i)$$

where $\lambda=\min\{k,m\}$.

These results are similar in form to those obtained in the scalar case, equation (III.77).

If a time invariant system is considered, then when the filter has reached steady state, the residual sequence following a fault has the following property:

$$E[\underline{\gamma}(k)\underline{\gamma}^T(m)] = H\Phi S_{\infty,\theta}(W^T)^{m-k}$$

where $S_{\infty,\theta}$ is the solution of the discrete time Lyapunov equation,

$$S_{\infty,\theta} = T + WS_{\infty,\theta}W^T$$

and, $T = KSK^T$

$$W = (I-KH)\Phi$$

The correlation matrix in the event of additional measurement noise is similar in form to equation (III.88) which describes the correlation in the event of additional measurement noise for the scalar system.

Thus, it is seen that the results concerning all types of faults, obtained in previous sections, may be carried over in a straightforward manner for the case of multivariable, time-varying systems.

CHAPTER IV: *FAULT MONITORING SCHEMES*

IV.1 General procedure for fault monitoring.

The results obtained for the joint pdf of residuals in the event of a fault occurrence, which are summarised in Section III.2.7, lead quite naturally to a hypothesis testing formulation of the fault monitoring process. Thus the hypothesis that the generated residuals belong to class C_0 against the hypothesis that it belongs to an alternative class will be tested.

As was asserted in Section II.4 the fault monitoring scheme must be designed in such a way as to be able to be applied in a wide range of practical situations with various requirements of cost and complexity. The knowledge of the effects of the individual faults on the Kalman filter innovations can be used to design a scheme that operates on two levels. The first level is a simple fault detection mechanism which also performs partial isolation of the failed parameter. On the sounding of an alarm from this first level, the second mechanism is activated. This performs the functions of fault isolation, estimation of time of occurrence and size of fault and subsequent system reorganization. Furthermore, if requirements so dictate, the two levels can be used independently of each other.

IV.1.1 General comments on hypothesis testing.

To test any hypothesis on a basis of a random sample of observations, the *sample space* Ω(i.e. all the possible sets of observations) is divided into two regions. If the observed point, say γ, falls into one of these regions, say ω, the hypothesis is rejected in favour of an alternative hypothesis; if γ falls into the complementary region $\Omega-\omega$ the hypothesis is accepted. ω is known as the *critical region* of the tests and $\Omega-\omega$ is called the *acceptance region*.

When making statistical hypothesis tests, the possibility of erroneous inference exists. This falls into two categories for the case where a null hypothesis H_o is tested against an alternative hypothesis H_1:

Type I: H_o is rejected when it is true.

Type II: H_o is accepted when it is false.

The probability of a type I error is equal to the size of the critical region used, termed the *significance level* of the test and denoted by α. Thus,

$$P[\gamma\epsilon\omega \mid H_o] = \alpha$$

In the present context α will be defined as the probability, P_f, of a false alarm. Hence,

$$P[\gamma\epsilon\omega \mid H_o] = P_f \qquad \text{(IV.1)}$$

The probability of a type II error is a function of the alternative hypothesis H_1, termed the *operating characteristic* of the test and denoted by β. Hence,

$$P[\gamma\epsilon\Omega-\omega \mid H_1] = \beta$$

$$P[\gamma\epsilon\omega \mid H_1] = 1-\beta$$

The complementary probability $1-\beta$ is called the *power* of the test and in the present context it will be defined as the probability, P_d, of

correct detection. Thus,

$$P[\gamma \varepsilon \omega | H_1] = P_d \qquad \text{(IV.2)}$$

For a given P_f, solution of (IV.1) will generally yield an infinity of subregions all obeying (IV.1). In this case ω is chosen so that P_d is maximum. This is a fundamental principle in statistical decision theory first expressed by J. Neyman and E. S. Pearson.

A critical region whose power is no smaller than that of any other region of the same size for testing a hypothesis H_o against an alternative H_1, is called a *best critical region* (BCR), and a test based on a BCR is called a *most powerful* (MP) test.

When testing a hypothesis H_o against a class of alternatives, i.e. a *composite* hypothesis (for example, when testing for a zero mean against nonzero mean) a MP test could be found for the different members of H_1 (an infinity for the aforementioned example). If there exists a BCR which is best for every member of H_1 then this region is called *uniformly most powerful* (UMP) and the test based on it a UMP test.

IV.1.2 Simple detection-partial isolation algorithms.

The first stage of the fault monitoring process involves fault detection. At this level, a decision is made, at each measurement update, of whether a fault is present or not. Using the results of Table 2, it can be seen that simple statistical tests for the mean and correlation properties of random variables may be used to decide if a fault has occurred or not. Further the disjoint nature of the three classes C_o, C_1, C_2 permits partial isolation of the fault. If C_1 and C_2 contain one element the isolation is unique but otherwise further isolation is needed. A decision flowchart for the first stage is shown in fig.5.

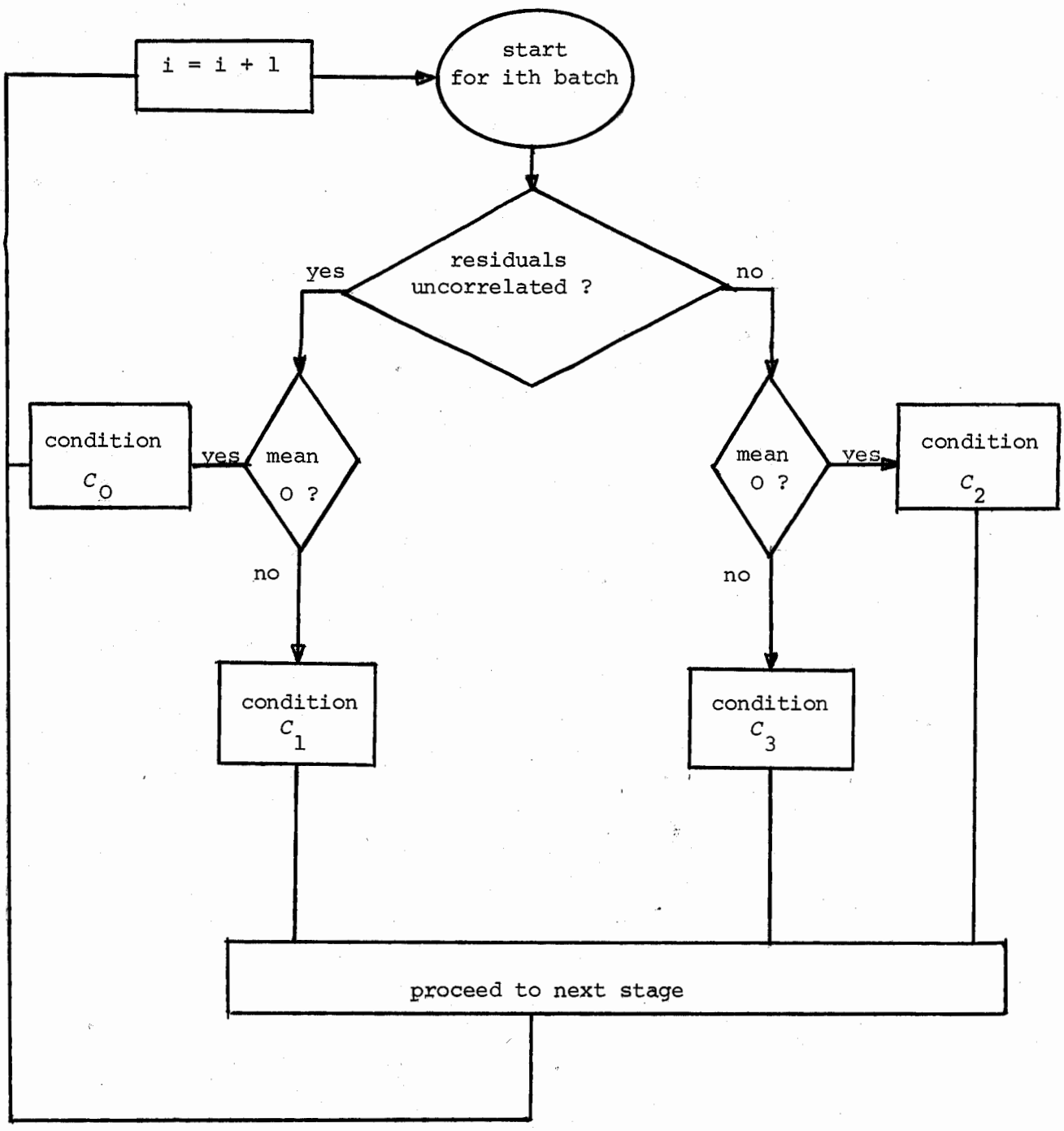

Fig. 5 The detection-partial isolation stage.

Decisions regarding the property of whiteness and zero mean will be tested sequentially using the concept of *Control charts*. These will be used to check the properties of whiteness and zero mean and a decision will be made at every time according to fig. 5.

IV.1.2.1 Control charts.

Control charts are used in monitoring the statistical state of a process whose measurements are available sequentially in time. Some statistic w(sample mean or sample range etc.) is computed from successive samples of size n and plotted on a graph containing lower and upper limits corresponding to the critical region of the hypothesis on w under test. If the statistic w is distributed normally with mean m_w and variance s_w^2, where m_w and s_w are calculated a-priori, then typical limits are m_w+3s_w for the *upper control limit* (UCL) and m_w-3s_w for the *lower control limit* (LCL). Such charts are usually referred to as *Shewhart charts* (fig. 6).

Control charts will be used for the first stage of fault monitoring as follows: given successive samples of residuals $\underline{\gamma}^{i,j}$, $\underline{\gamma}^{i+1,j+1}$, ..., $\underline{\gamma}^{i+m,j+m}$ an appropriate statistic will be calculated and plotted on a corresponding control chart with the precomputed UCL and LCL. A decision that a fault has occurred will be made when the statistic falls outside its normal operation level for a specified subsequent number of times. This procedure will decrease the probability of type I errors. This may be seen in the following argument:

Let d = 1 if a decision that a fault has occurred is made

d = 0 if otherwise

Then d is distributed binomially and since the probability of a type I error is α at every point, the probability that a type I error is made at m successive times is α^m. A logical flowchart for a computer-operated control chart is shown in fig. 7.

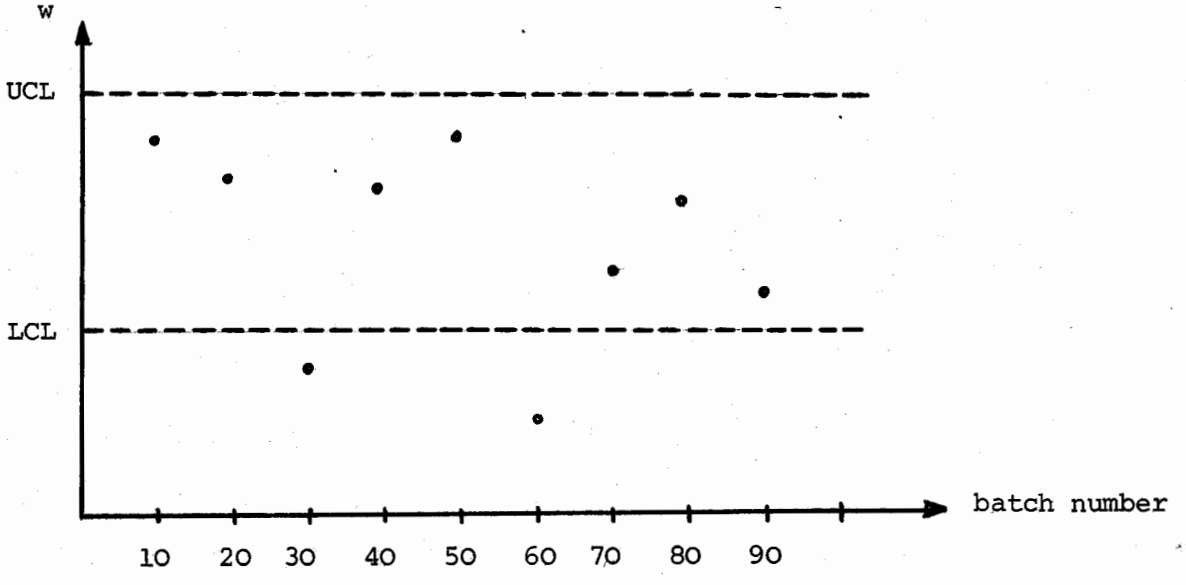

Fig. 6 Shewhart control chart

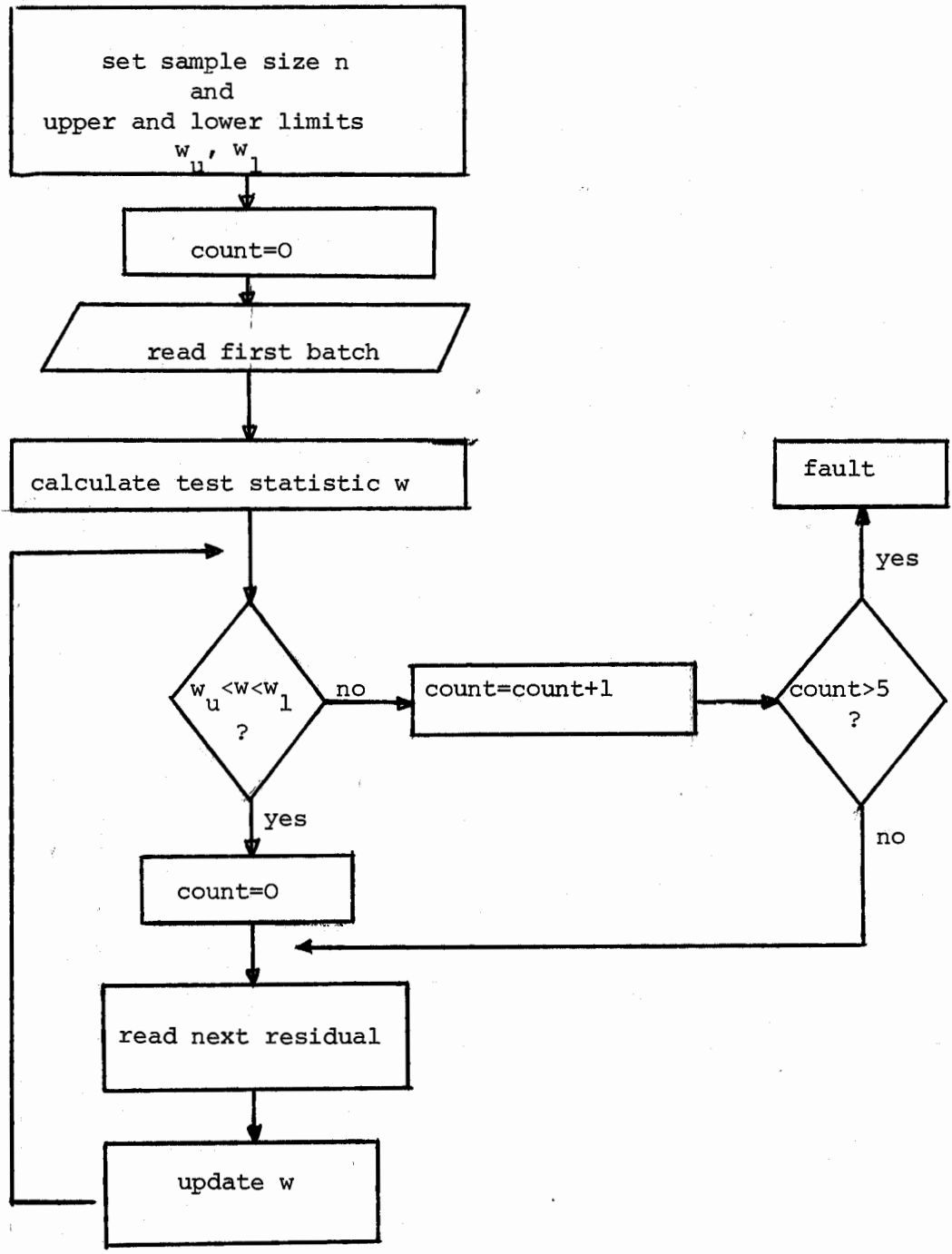

Fig. 7 Flowchart for computer operated control chart.

IV.1.2.1.1 Testing for the mean.

If class C_3 is included as a possible fault case, the mean of the residual sequence will have to be calculated from a series of correlated measurements, if such a fault occurs. If C_3 is excluded, the sample mean will be used only for uncorrelated measurements. This can be accomplished by testing first for independence and then for the mean. If the independence test is positive C_2 is selected, otherwise the mean is tested to decide between C_1 and C_0; both of these classes contain independent residuals.

Tests for both situations will be presented.

a. *Sample mean.*

The test statistic commonly used for testing,

$$H_0: \bar{\gamma}(k) = 0$$

against $H_1: \bar{\gamma}(k) = \bar{\gamma}_1(k) \neq 0 \; ; \quad k=i,\ldots,j$

is the sample mean defined by:

$$\hat{\bar{\gamma}} = \frac{1}{n} \sum_{k=i}^{j} \gamma(k) \qquad \text{(IV.3)}$$

Under the null hypothesis, the sample mean is normally distributed with zero mean and variance c/n, where c is the steady state value of the innovations sequence and n is the size of the sample.

The probabilities P_f and P_d are respectively given by:

$$P_f = P\left[\, |\hat{\bar{\gamma}}| > \sqrt{\frac{c}{n}}\, z_{P_f/2} \right] \qquad \text{(IV.4)}$$

and

$$P_d = 1 - \left\{ \Phi\left[-\bar{\gamma}_1(k)\sqrt{\frac{n}{c}} + z_{P_f/2}\right] - \Phi\left[-\bar{\gamma}_1(k)\sqrt{\frac{n}{c}} - z_{P_f/2}\right] \right\} \qquad \text{(IV.5)}$$

where z_a is defined by:

$$P[z>z_a] \triangleq \frac{1}{\sqrt{2\pi}} \int_{z_a}^{\infty} e^{-\frac{1}{2}z^2} dz = a \quad \text{(pdf of the standard normal didstribution)}$$

and $$\Phi(z) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}y^2} dy \quad \text{(cumulative df of the standard normal distribution)}$$

P_f, P_d and n are functionally related in the two equations defining P_f and P_d. P_d also depends on the unknown value $\bar{\gamma}(k)$. Typical values for P_f are 0.1, 0.05 though this will of course depend on the specific application requirements. Having fixed P_f, then P_d, n and the critical region $|\hat{\gamma}|$ can be chosen using equations (IV.4)-(IV.5).

The UCL and LCL values are given by:

$$\left.\begin{aligned} UCL &= \frac{\sqrt{c}}{n} z_{P_f/2}, \\ LCL &= -\frac{\sqrt{c}}{n} z_{P_f/2} \end{aligned}\right\} \quad \text{where } z_{P_f/2} \text{ is determined from the standard normal distribution given } P_f$$

The graph of $1-P_d$, called the *operating characteristic* (OC) curve is shown in figs. 8 and 9 for different values of the sample size n and for P_f=0.05 and 0.01 respectively. As it can be seen from the graphs, increasing the sample size increases P_d, but at the expense of an increase in the detection delay time, since by averaging a larger number of residuals the effect of a fault is smoothed out.

The sample mean $\hat{\gamma}$ can be calculated iteratively, thus reducing the amount of computation in on-line operations. Define,

$$\hat{\gamma}_{i,j} \triangleq \frac{1}{n} \sum_{k=i}^{j} \gamma(k)$$

then,

$$\hat{\gamma}_{i+1,j+1} = \frac{1}{n} \sum_{k=i+1}^{j+1} \gamma(k)$$

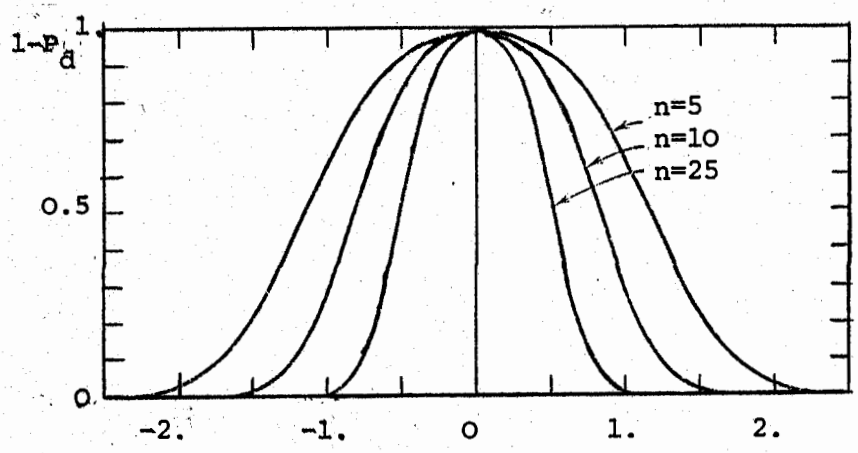

Fig. 8 P_f=0.01

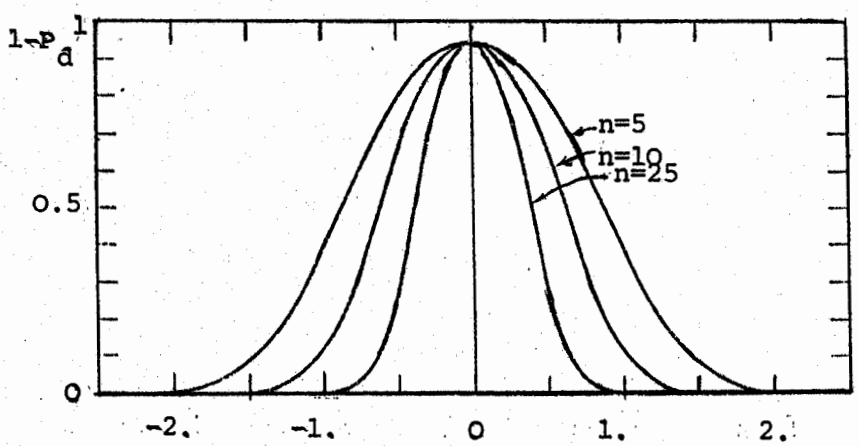

Fig. 9 P_f=0.05

Figs. 8,9 Operating characteristic curves for the sample mean test plotted against true mean/variance.

$$= \frac{1}{n}\Big\{ \sum_{k=i}^{j} \gamma(k) + \gamma(j+1) - \gamma(i) \Big\}$$

$$= \hat{\gamma}^{i,j} + \frac{1}{n}\{\gamma(j+1) - \gamma(i)\}$$

If the residuals are correlated, the sample mean test may still be used but its control limits have to be modified accordingly. Statistical tests for the mean in the presence of correlated measurements do not appear to exist in the statistical literature. This means that in such cases the robustness of the appropriate tests must be examined when the assumption of independence is violated.

To calculate the effect on the sample mean control limits, consider the variance of the residual sample mean, which is now calculated using the formula *[25]*:

$$\text{var}[\hat{\bar{\gamma}}] = \frac{1}{n^2}\{nc + 2(n-1)c\rho_1 + 2(n-2)c\rho_2 + \ldots$$

where ρ_i is the ith order correlation between the residuals and c is the steady state variance of the residuals under the null hypothesis. Then,

$$\text{var}[\hat{\bar{\gamma}}] = \frac{c}{n} + \frac{2c}{n^2} \sum_{k=1}^{n-1} (n-k)\rho_k$$

$$\simeq \frac{c}{n}\Big\{1 + 2 \sum_{k=1}^{\infty} \rho_k\Big\}$$

If the process is autoregressive of order 1, then $\rho_k = \rho^k$. Hence, for such processes,

$$\sum_{k=1}^{\infty} \rho_k = \sum_{k=1}^{\infty} \rho^k = \frac{\rho}{1-\rho}$$

Consequently,

$$\text{var}[\hat{\bar{\gamma}}] \simeq \frac{c}{n}\left\{1 + \frac{2\rho}{1-\rho}\right\} = \frac{c}{n}\,\frac{1+\rho}{1-\rho} \qquad \text{(IV.6)}$$

This result implies that in the case of correlated measurements, the limits of the control chart for the sample mean have to be modified according to (IV.6). If the correlation is negative the limits have to be decreased, whereas if the correlation is positive the limits have to be increased, since,

$$\frac{1+\rho}{1-\rho} < 1 \text{ if } \rho<0$$
$$> 1 \text{ if } \rho>0$$

In the first stage of the fault monitoring process the correlation is not known, therefore if the occurred fault induces large ρ the mean test will give erroneous results.

b. *Sign test.*

This is a non-parametric test used to test hypotheses on the value of the median of a population. Since the residuals are normal under all hypotheses the median is equal to the mean and therefore this test can be applied to test for zero mean.

The sign test procedure is as follows: the number of positive residuals in a batch is calculated and compared to two thresholds which depend on the sample size n and significance level α. Thus if,

$$n_1 < \text{(number of positive residuals)} < n_2; \quad \text{accept } H_0$$

otherwise H_0 is rejected.

Table 4 is a table of the percentage points of the symmetric binomial distribution for different sample sizes and significance levels. It is shown in [51] that it may be used for the sign test as follows:

i. Count the number of values above and below zero, say n^+ and n^-.

ii. Choose the smallest of the two values, say n^+.

TABLE 4

Percentage points of the symmetric binomial distribution.

Sample size	Probability of false detection			
n	P_f=0.10	P_f=0.05	P_f=0.02	P_f=0.01
5	1	-	-	-
6	1	1	-	-
7	1	1	1	-
8	1	1	1	1
9	2	2	1	1
10	2	2	1	1
11	3	2	2	1
12	3	3	2	2
13	4	3	2	2
14	4	3	3	2
15	4	4	3	3
16	5	4	4	3
17	5	5	4	3
18	6	5	4	4
19	6	5	5	4
20	6	6	5	4
21	7	6	5	5
22	7	6	6	5
23	8	7	6	5
24	8	7	6	6
25	8	8	7	6
30	11	10	9	8
35	13	12	11	10
40	15	14	13	12
45	17	16	15	14
50	19	18	17	16

iii. Compare n^+ with the table entry for chosen n and α, say n_α.

iv. If $n^+ < n_\alpha$, reject H_o; otherwise accept it.

The entries in Table 4 may be modified to indicate percentage points for the number of positive residuals. If for a sample size n the table entry is n_α, it follows that the number of positive residuals can vary from n_α to $n-n_\alpha$. The thresholds n_1 and n_2 are then chosen to satisfy:

$$n_1 = n_\alpha$$
$$n_2 = n - n_\alpha$$

Hence n_1 and n_2 represent the UCL and LCL respectively.

Tables 5 and 6 show values of $1-P_d$ for P_f of 0.05 and 0.01 respectively *[56]*.

The number of positive residuals can also be calculated iteratively. Let $n^+_{i,j}$ be the number of positive residuals in the residual vector $\underline{\gamma}^{i,j}$ and,

$$n_i = 1 \text{ if } \gamma(i) > 0$$
$$= 0 \text{ if } \gamma(i) < 0$$

(the best procedure for residual values that are equal to zero is to disregard them and reduce the sample size by their number. This is also intuitively appealing since a zero value contributes equally to both negative and positive values).

Then,

$$n^+_{i+1,j+1} = n^+_{i,j} + n_{j+1} n_i$$

The robustness of the sign test in the case of correlated residuals, can be investigated similarly to the case of the sample mean test.

Let, $n_i = 1 \text{ if } \gamma(i) > 0$

$$= -1 \text{ if } \gamma(i) < 0$$

TABLES 5,6

Values of $1-P_d$ for the sign test

Table 5

$P_f = 0.05$

n	r	$1-P_d$
8	0	.00781
9	0	.00391
10	0	.00195
11	0	.00098
12	1	.00635
13	1	.00342
14	1	.00183
15	2	.00739
16	2	.00418
17	2	.00235
18	3	.00754
19	3	.00443
20	3	.00258
25	5	.00408
30	7	.00522
35	9	.00599
40	11	.00643
45	13	.00661
50	15	.00660
60	19	.00622
70	23	.00558
80	28	.00968
90	32	.00743
100	36	.00664

Table 6

$P_f = 0.01$

n	r	$1-P_d$
(5)	(0)	.0625
6	0	.03125
7	0	.01562
8	0	.00781
9	1	.03906
10	1	.02148
(10)	(2)	.10938
11	1	.01172
12	2	.03857
13	2	.02246
14	2	.01294
15	3	.03516
16	3	.02127
17	4	.04904
18	4	.03088
19	4	.01921
(20)	(4)	.01182
20	5	.04139
(20)	(6)	.11532
25	7	.04329
30	9	.04277
35	11	.04096
40	13	.03848
45	15	.03570
50	17	.03284
60	20	.02734
70	26	.04139
80	30	.03299
90	35	.04460
100	39	.0352

Then, $E[n_i] = 0$

$\text{var}[n_i] = 1$

The random variable n_i may be associated with the positive and negative residuals. Hence, if the $\gamma(i)$ are correlated,

$$\text{cov}[n_i, n_{i+j}] = E[n_i n_{i+j}]$$

$$= \frac{2}{\pi} \sin^{-1}\rho_j \quad [25]$$

where $\rho_j = E[\gamma(i)\gamma(i+j)]$

If $\rho_j = \rho^j$, $E[n_i n_{i+j}] = \frac{2}{\pi} \sin^{-1}\rho^j \triangleq \rho_j^{(s)}$

The variance of the sign test statistic will be given by:

$$\text{var}[n^+_{i,j}] = \frac{[\sigma^{(s)}]^2}{n} \left\{1 + \frac{2}{n} \sum_{h=1}^{n-1} (n-h)\rho_h^{(s)}\right\}$$

The sum is equal to,

$$\sum_{h=1}^{n-1} \frac{2}{\pi} \sin^{-1}\rho^h$$

Expanding the inverse sine in Taylor's series about 0,

$$\sin^{-1}\rho^h = \left(\frac{d}{d\rho} \sin^{-1}\rho^h\right)\Big|_{\rho=0} \rho^h + O(\rho^{2h})$$

$$= \rho^h + O(\rho^{2h})$$

Hence if orders >2 are neglected

$$\text{var}[n^+_{i,j}] = \frac{[\sigma^{(s)}]^2}{n} \left\{1 + \frac{2\rho}{\pi(1-\rho)}\right\}$$

$$= \frac{[\sigma^{(s)}]^2}{n} \quad \frac{\pi+\rho(2-\pi)}{\pi(1-\rho)} \qquad \text{(IV.7)}$$

The modifying factor in this case is appreciably less than the factor appearing in (IV.6). This suggests that the sign test will be more robust in departures from independence than the corresponding sample mean test.

IV.1.2.1.2 Testing for whiteness.

Testing for whiteness is a common requirement for a number of identification algorithms that appear in the control literature. Most methods however require a large sample size (>500) and are thus inapplicable to the case of on-line fault detection. These methods include: plotting of the sample autocorrelation coefficients *[35]*, *[52]*, hypothesis testing on the diagonal form of the correlation matrix *[53]*, Stoica's test *[54]* and others.

The large sample size requirement for the above tests is due to the fact that the probability distributions of the statistics used cannot be found for small sample sizes and therefore approximations to the normal distribution have to be made by using an appropriate sample size.

Three tests for whiteness were investigated. These include two parametric and one non-parametric:

i. First order serial correlation, r_1.

ii. Sample variance $\hat{s}^2$.

iii. Rank correlation.

The test on the variance is included since, as it can be seen from Table 3, the assumed variance is the correct one, only when the residuals are white. Error in the variance therefore implies non-whiteness.

All of these tests are well documented in the statistical literature and a brief review of each method is given here.

a. *First order serial correlation.*

The first order serial correlation is defined by

$$r_1 = \frac{\sum_{m=i}^{j-1} \{(\gamma(m) - \hat{\gamma}^{i,j})(\gamma(m+1) - \hat{\gamma}^{i,j})\}}{\sum_{m=i}^{j} (\gamma(m) - \hat{\gamma}^{i,j})^2}$$

This form is used for mathematical and computational convenience. For small sample sizes (<20) more accurate forms may be used *[25]*. The statistic r_1 is distributed asymptotically normal with mean $-1/n-1$ and variance $1/n$. However it is shown in *[55]*, that r_1 is normally distributed for small sample sizes as well (n=10).

Thus confidence limits for hypothesis testing can be found in a similar way to that used for the sample mean.

b. *Sample variance.*

The variance is calculated from the sample by the formula,

$$\hat{s}^2 = \frac{1}{n} \sum_{m=i}^{j} (\gamma(m) - \hat{\gamma}^{i,j})^2$$

Confidence limits for testing,

$$H_0: \sigma^2 = c$$

against $$H_1: \sigma^2 \neq c = \sigma_1^2$$

are found using the fact that the quantity

$$\chi^2_{n-1} = \frac{(n-1)\hat{s}^2}{\sigma^2}$$

is distributed χ^2 with (n-1) degrees of freedom. It then follows that the relation

$$\frac{(n-1)\hat{s}^2}{\chi^2_{n-1,\frac{1}{2}\alpha}} < c < \frac{(n-1)\hat{s}^2}{\chi^2_{n-1,1-\frac{1}{2}\alpha}}$$

will have a probability of $1-\alpha$ of being correct *[51]*. Equivalently,

$$\frac{\chi^2_{n-1,1-\frac{1}{2}\alpha^c}}{n-1} < \hat{s}^2 < \frac{\chi^2_{n-1,\frac{1}{2}\alpha^c}}{n-1}$$

represent the confidence limits on $\hat{s}^2$ with a probability of type I error α. Hence, the UCL and LCL are given by:

$$UCL = \frac{\chi^2_{n-1,\frac{1}{2}\alpha^c}}{n-1} \qquad (IV.8)$$

$$LCL = \frac{\chi^2_{n-1,1-\frac{1}{2}\alpha^c}}{n-1} \qquad (IV.9)$$

The power of the test is given by:

$$\pi(\sigma_1^2) = P[\chi^2_{n-1} < \frac{1}{\lambda^2}\chi^2_{n-1,\frac{1}{2}\alpha}] + P[\chi^2_{n-1} > \frac{1}{\lambda^2}\chi^2_{n-1,1-\frac{1}{2}\alpha}] \qquad (IV.10)$$

where $\lambda^2 = \frac{\sigma_1^2}{c}$.

Fig. 10 shows some power curves for P_f=0.05 and n=3, 10, 30.

The variance and the first order serial correlation, as indeed correlations of higher order can be calculated iteratively. The equations describing the evolution of correlations are developed in Appendix I.3†. These are:

$$\hat{c}_o^j = \hat{c}_o^{j-1} + \frac{1}{n}\{\gamma^2(j)-\gamma^2(i-1)\} - a^j(a^j+2\hat{\gamma}^{i-1,j-1})$$

$$\hat{\gamma}^{i,j} = \hat{\gamma}^{i-1,j-1} + a^j$$

$$a^j = \frac{1}{n}\{\gamma(j)-\gamma(i-1)\}$$

and c_m^j can be calculated from c_{m-1}^j using:

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† These results have also been submitted in the form of a Technical note paper for publication in IEEE Trans. on Aut. Control.

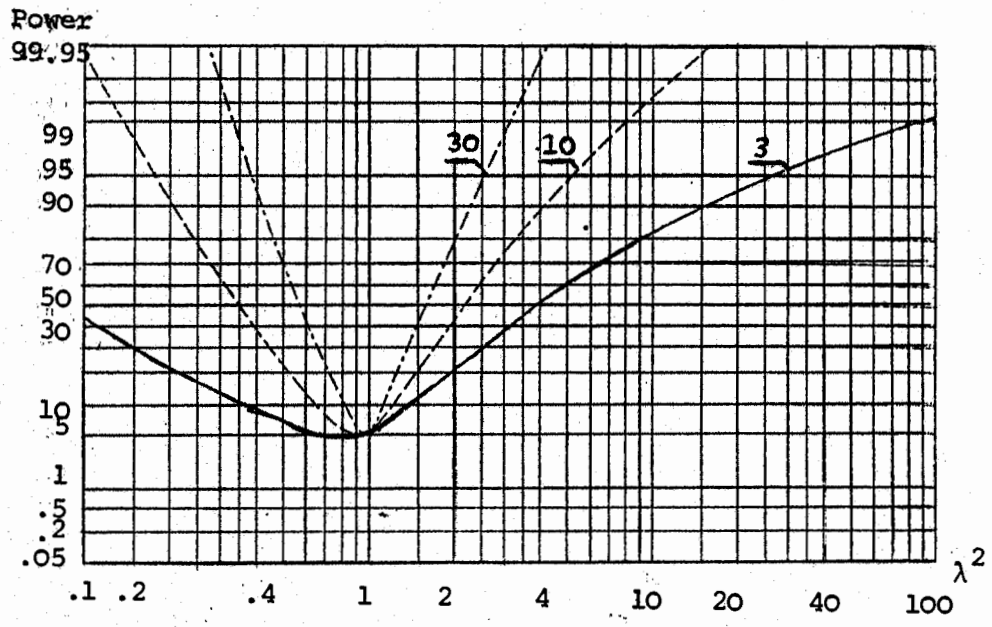


Fig. 10 Power curves for the two-tailed χ^2-test at the 5% level of significance.

$\lambda^2=\sigma^2/\sigma_0^2$

$$\hat{c}_m^j = \frac{1}{n}\left\{(n-m)\left[\hat{\gamma}^{i,j}\right]^2 - (q_m^j+p_m^j)\hat{\gamma}^{i,j} + \sum_{k=k_1}^{j}\gamma(k)\gamma(k-m)\right\}$$

where,

$$q_m^j = q_{m-1}^j - \gamma(j-n-m)$$

$$p_m^j = p_{m-1}^j - \gamma(j-n+1)$$

$$p_o^j = q_o^j = n\hat{\gamma}^{i,j}$$

and $\hat{c}_m^j$ denotes the sample serial correlation of lag m calculated from the residual sample $\underline{\gamma}^{i,j}$. Specifically the variance, $\hat{s}^2=\hat{c}_o^j$, and the first order serial correlation $r_1=\hat{c}_1^j$ can be calculated iteratively using the above formulae.

c. *Rank correlation*.

The rank correlation coefficient is the non-parametric equivalent to the standard correlation coefficient of two sets of variables. The usual procedure for the calculation of the rank correlation coefficient for a set of values $x_1,y_1;\ x_2,y_2;\ldots;\ x_n,y_n$, is to replace each x_j, y_j by their rankings x_j' and y_j' among the x's and y's respectively, and calculate:

$$r' = 1 - \frac{\sum_{j=1}^{n} d_j^2}{n(n^2-1)}$$

where $d_j = x_j'-y_j'$.

If serial independence is to be tested, the set $\{x_j\}$ is replaced by the set $\{j\}$, while the set $\{y_j\}$ represents the population values.

Therefore, to test for whiteness of the residual sequence, calculate,

$$r' = \frac{\sum_{m=i}^{j}\{m-i+1-\gamma'(m)\}^2}{n(n^2-1)}$$

where $\gamma'(m)$ is the rank of $\gamma(m)$ among the $\gamma(i)$'s.

The calculated value of r' is then compared to its LCL and UCL values which are found from Table 7, for different P_f.

IV.1.3 Detection-isolation-estimation algorithms.

The algorithms described in this section may be used to provide all three functions of the monitoring process. In this case, the computational time will increase, as the statistical calculations become more complicated, but overall system performance will be improved. as a result of simultaneous detection, isolation and identification. Alternatively, the algorithms may be used in conjunction with the simpler detection-partial isolation algorithms, in which case the complexity will be introduced after the detection of a fault.

The statistical procedures for this stage, depend on *maximum likelihood estimation* (MLE) and particularly on the *generalised likelihood ratio* (GLR) test. A brief description of these procedures will now be given.

IV.1.3.1 Likelihood ratio tests, MLE and GLR tests.

Likelihood ratio (LR) tests are used in hypothesis testing when all parameters of the pdf's under both hypotheses are known. Then the LR test statistic is defined as:

$$\Lambda \triangleq \frac{p(\underline{x}|H_1)}{p(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

where $p(\underline{x}|H_i)$ is the pdf of $\underline{x}$ under H_i and λ is a threshold value chosen to give required values of P_d and P_f. This is usually accomplished by fixing $P_f=\alpha$, and choosing λ to maximise P_d, by solving

TABLE 7

Critical values of the rank correlation coefficient

n \ P_f	0.1	0.05	0.02	0.01
5	0.9	-	-	-
6	0.829	0.886	0.943	-
7	0.714	0.786	0.893	-
8	0.643	0.738	0.833	0.881
9	0.6	0.683	0.783	0.833
10	0.564	0.648	0.745	0.794
11	0.523	0.623	0.736	0.818
12	0.497	0.591	0.703	0.78
13	0.475	0.566	0.673	0.745
14	0.457	0.545	0.646	0.716
15	0.441	0.525	0.623	0.689
16	0.425	0.507	0.601	0.666
17	0.412	0.49	0.582	0.645
18	0.399	0.476	0.564	0.625
19	0.388	0.462	0.549	0.608
20	0.377	0.45	0.534	0.591
21	0.368	0.438	0.521	0.576
22	0.359	0.428	0.508	0.562
23	0.351	0.418	0.496	0.549
24	0.343	0.409	0.485	0.537
25	0.336	0.4	0.475	0.526
26	0.329	0.392	0.465	0.515
27	0.323	0.385	0.456	0.505
28	0.317	0.377	0.448	0.496
29	0.311	0.37	0.44	0.487
30	0.305	0.364	0.432	0.478

for λ,

$$\int_{\lambda}^{\infty} p(\Lambda|H_0)\,d\Lambda = \alpha, \quad [44].$$

If samples of n independent observations which have the same distribution are made, the joint probability of the observations regarded as a function of an unknown parameter $\underline{\xi}$ is called the *likelihood function* (LF) of the sample and is written:

$$L(\underline{x};\underline{\xi}) = f(\underline{x}_1;\underline{\xi})\,f(\underline{x}_2;\underline{\xi})\ldots f(\underline{x}_n;\underline{\xi})$$

The MLE $\hat{\underline{\xi}}$ is the value of $\underline{\xi}$, within the admissible range of values, which maximises $L(\underline{x};\underline{\xi})$. That is $\hat{\underline{\xi}}$ satisfies,

$$L(\underline{x};\hat{\underline{\xi}}) \geq L(\underline{x};\underline{\xi})$$

If the LF is a twice differentiable function of $\underline{\xi}$, throughout its range, stationary values of the LF, where they exist, will be given by the roots of,

$$L'(\underline{x};\underline{\xi}) = \frac{\partial L(\underline{x};\underline{\xi})}{\partial \underline{\xi}} = 0$$

A sufficient , though not necessary, condition that any of these stationary values, say $\tilde{\underline{\xi}}$, be a local maximum is that,

$$L''(\underline{x};\tilde{\underline{\xi}}) < 0$$

In practice, it is often simpler to work with the logarithm of the LF than with the function itself. Under the above conditions they will have maxima together, since

$$\frac{\partial \log L}{\partial \underline{\xi}} = \frac{L'}{L}$$

and L>0. Therefore solutions of,

$$(\log L)' = 0$$

are sought, for which,

$$(\log L)'' < 0.$$

ML estimators may be used in hypothesis-testing where one or both of the hypotheses contain an unknown random or non-random parameter. A procedure commonly employed in this case is the GLR test. It consists of calculating the ML estimate of the unknown parameter under both hypotheses, and then using these values to form a likelihood ratio test. The value of the LR is then compared to a given threshold and a decision made depending on whether the LR is above or below the threshold.

Let the joint pdf of the sample under H_i be $p(\underline{x}|H_i;\underline{\xi})$, and

$$\max_{\xi \epsilon \Xi_i} p(\underline{x}|H_i;\underline{\xi}) = p(\underline{x}|H_i;\hat{\underline{\xi}}_i)$$

The GLR statistic is then calculated by,

$$\Lambda_g = \frac{p(\underline{x}|H_1;\hat{\underline{\xi}}_1)}{p(\underline{x}\ H_0;\hat{\underline{\xi}}_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

or equivalently,

$$\ln\Lambda_g = \ln p(\underline{x}|H_1;\hat{\underline{\xi}}_1) - \ln p(\underline{x}|H_0;\hat{\underline{\xi}}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\lambda$$

It is in general difficult to obtain the values of P_d and P_f for the GLR test because of the complexity of the probability distribution of its statistic. However, bounds on P_d and P_f can be calculated using approximate formulae for the distribution of Λ_g *[44]*. Alternatively, simulation methods, using Monte Carlo techniques may be used for the evaluation of P_d and P_f.

In theory, this procedure may be extended in situations where the alternative hypotheses are more than one. In this case the decision space is divided into hyperplanes, each of which corresponds to given hypothesis. This idea is illustrated in fig. 11. for the case of three hypotheses H_o, H_1, H_2. The plane is divided into three regions and the point $(\log\Lambda_g^1, \log\Lambda_g^2)$, where Λ_g^1, Λ_g^2 denote the GLR test values of H_o against H_1 and H_2 respectively, is used to decide which of the hypotheses is more likely. In tests where the total probability of error is to be minimised, a sensible procedure is to choose the hypothesis with the largest a-posteriori probability value *[44]*.

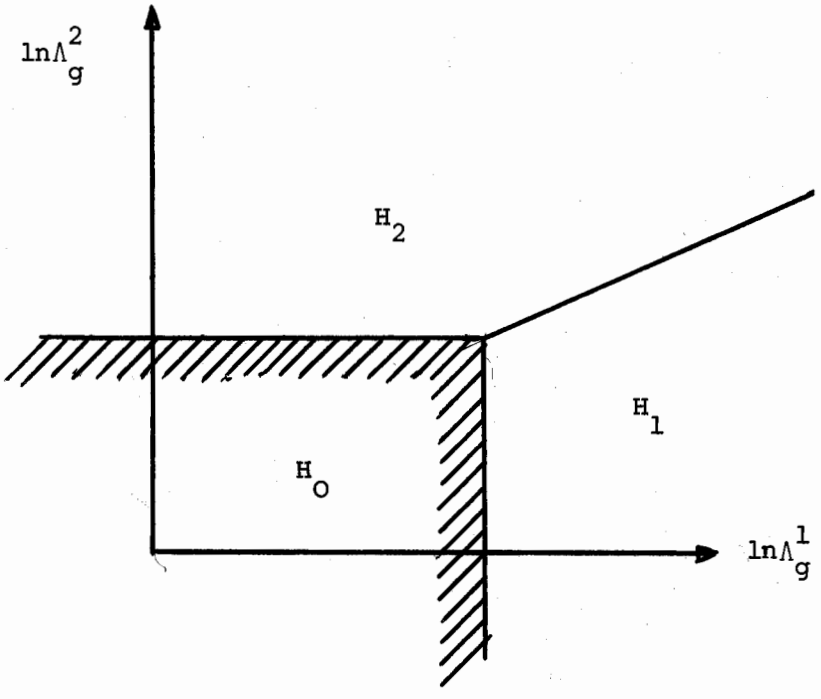


Fig. 11 Multiple hypothesis decision regions.

IV.1.3.2 Null and alternative hypotheses for the GLR test.

In Section III.1.2.it was shown that the Kalman filter residuals may be modelled as,

$$\gamma(k) = \gamma_0(k) + g(k,\underline{\xi})$$

for any system condition. If no fault occurs, $g(k,\underline{\xi}) \equiv 0$, and hence,

$$\gamma(k) = \gamma_0(k)$$

Here, $\underline{\xi}$ is an unknown vector whose components are the time of the occurrence of the fault and its magnitude. Also, in all the cases of interest,

$$g(k,\underline{\xi}) = g(k,\theta)\Delta p$$

where Δp is the fault size and θ the time of fault occurrence.

Therefore, in cases where two alternatives exist, the two hypotheses may be written:

$$H_0 : \gamma(k) = \gamma_0(k)$$
$$H_1 : \gamma(k) = \gamma_{0(k)} + g(k,\underline{\xi})$$

Since only H_1 contains $\underline{\xi}$, the GLR test is:

$$\Lambda_g = \frac{p(\underline{\gamma}^{i,j}|H_1;\hat{\underline{\xi}})}{p(\underline{\gamma}^{i,j}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

If the GLR tests are used in conjunction with the simple detection partial isolation algorithms, both hypotheses on the test will be hypotheses of faults. There can be two possible pairs, if the monitoring is restricted to the parameters of the noise sequences:

1. H_0: $\gamma(k) = \gamma_0(k) + g_a(k,\underline{\xi})$

   H_1: $\gamma(k) = \gamma_0(k) + g_d(k,\underline{\xi})$

2. H_0: $\gamma(k) = \gamma_0(k) + g_c(k,\underline{\xi})$

   H_1: $\gamma(k) = \gamma_0(k) + g_f(k,\underline{\xi})$

The first pair of hypotheses will be tested if the occurred fault is a bias in the state or measurement while the second pair will be tested if the occurred fault is additional plant or measurement noise.

In both pairs the GLR statistic will be of the form:

$$\Lambda_g = \frac{p(\underline{\gamma}|H_1;\hat{\underline{\xi}}_1)}{p(\underline{\gamma}|H_0;\hat{\underline{\xi}}_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

GLR tests whose null and alternative hypotheses are hypotheses of fault will be called *composite*, otherwise they will be called *simple*. Both composite and simple GLR involve the maximisation, with respect to the unknown parameters, of the joint pdf of the residual sequence under faulty conditions. Since the residuals can be either of zero mean and correlated or of nonzero mean and independent, the maximisation of the pdf under these two conditions will be investigated.

IV.1.3.3 Maximisation of the joint pdf of residuals in fault conditions (a) or (d).

The joint pdf of the residuals in the case of a step bias of size ν at time θ in the state or the measurements is given from (III.66), (III.82), by,

$$p(\underline{\gamma}^{j,k}|H_1;\theta,\nu) = \pi(j,\theta-1)\prod_{i=\theta}^{k}\frac{1}{(2\pi c(i,i))^{\frac{1}{2}}}\exp\left\{-\tfrac{1}{2}\frac{\{\gamma(i)-\sum_{m=\theta}^{i}g(i,m)\nu\}^2}{c(i,i)}\right\}$$

where g stands for either g_a or g_d which are given by (III.22)-(III.24) and (III.42)-(III.45) respectively, and ν for either ν_x or ν_y.

Taking logarithms,

$$\ln p(\underline{\gamma}^{j,k}|H_1;\theta,\nu) = \Sigma(j,\theta-1) - \tfrac{1}{2}\sum_{m=\theta}^{k}\frac{\{\gamma(m)-\sum_{i=\theta}^{m}g(m,i)\nu\}^2}{c(m,m)} \qquad \text{(IV.11)}$$

where,

$$\Sigma(j,\theta-1) = -\tfrac{1}{2}\left\{\sum_{m=j}^{k}\ln\{2\pi c(m,m)\} - \sum_{m=j}^{\theta-1}\frac{\gamma^2(m)}{c(m,m)}\right\} \qquad \text{(IV.12)}$$

The range of values of the unknown parameters θ and ν are defined by the following sets:

$$\Theta = \{\theta: j<\theta<k;\ \theta\varepsilon N^+\} \qquad \text{(IV.13)}$$

$$N = \{\nu: \nu_l<\nu<\nu_u:\ \nu\varepsilon R\} \qquad \text{(IV.14)}$$

Since the first term on the RHS of (IV.I2) is independent of θ, the following maximisation is required:

$$\max_{\substack{\theta\varepsilon\Theta\\ \nu\varepsilon N}}\left\{-\tfrac{1}{2}\left\{\sum_{m=j}^{\theta-1}\frac{\gamma^2(m)}{c(m,m)} + \sum_{m=\theta}^{k}\frac{\{\gamma(m)-\sum_{i=\theta}^{m}g(m,i)\nu\}^2}{c(m,m)}\right\}\right\} \qquad \text{(IV.15)}$$

Alternatively, the minimum of the negative of (IV.15) can be found. At a minimum, partial derivatives vanish, hence solve first for ν,

$$\frac{\partial}{\partial\nu}\left\{\sum_{m=j}^{\theta-1}\frac{\gamma^2(m)}{c(m,m)} + \sum_{m=\theta}^{k}\frac{\{\gamma(m)-a(m)\nu\}^2}{c(m,m)}\right\}$$

$$= -2\sum_{m=\theta}^{k}\frac{a(m)\{\gamma(m)-a(m)\nu\}}{c(m,m)} = 0$$

where $a(m) = \sum_{i=\theta}^{m} g(m,i)$

This vanishes if,

$$\nu = \frac{\sum_{m=\theta}^{k} \frac{a(m)\gamma(m)}{c(m,m)}}{\sum_{m=\theta}^{k} \frac{a^2(m)}{c(m,m)}} \qquad \text{(IV.16)}$$

Substituting (IV.16) into (IV.15) yields an expression for the optimum θ:

$$\max_{\theta\in\Theta} \left\{ \sum_{m=j}^{k} \frac{\gamma^2(m)}{c(m,m)} - 2 \frac{\left\{ \sum_{m=\theta}^{k} \frac{a(m)\gamma(m)}{c(m,m)} \right\}^2}{\sum_{m=\theta}^{k} \frac{a^2(m)}{c(m,m)}} \right\} \qquad \text{(IV.17)}$$

The first term is independent of θ, hence $\hat{\theta}$ is the value of θ that maximises the second term in (IV.17). If $\hat{\theta}$ is substituted in (IV.16) $\hat{\nu}$ is obtained. Finally $\hat{\theta}$ and $\hat{\nu}$ are substituted in (IV.11) to yield the desired expression :

$$\ln p(\underline{\gamma}^{j,k}|H_1,\hat{\theta},\hat{\nu}) = -\tfrac{1}{2}\left\{ \sum_{m=j}^{k} \ln 2\pi c(m,m) + \frac{\gamma^2(m)}{c(m,m)} \right\} + \frac{\left\{ \sum_{m=\hat{\theta}}^{k} \frac{a(m)\gamma(m)}{c(m,m)} \right\}^2}{\sum_{m=\hat{\theta}}^{k} \frac{a^2(m)}{c(m,m)}} \qquad \text{(IV.18)}$$

The logarithm of the GLR test statistic will then be:

$$\ln \Lambda_g = \ln p(\underline{\gamma}^{j,k}|H_1,\hat{\theta},\hat{\nu}) - \ln p(\underline{\gamma}^{j,k}|H_0) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\lambda$$

$$= \frac{\left\{ \sum_{\hat{\theta}}^{k} \frac{a(m)\gamma(m)}{c(m,m)} \right\}^2}{\sum_{m=\hat{\theta}}^{k} \frac{a^2(m)}{c(m,m)}} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\lambda \qquad \text{(IV.19)}$$

IV.1.3.4 Maximisation of the joint pdf of residuals in cases (c) or (f).

The joint pdf of the residual sequence in the case of additional plant or measurement noise is given from (III.79), (III.90) by;

$$p(\underline{\gamma}^{j,k}|H_1,\theta,s) = \pi(j,\theta-1)\frac{1}{2\pi^{\frac{1}{2}n}|C|^{\frac{1}{2}}}\exp\{-\tfrac{1}{2}\,\underline{\gamma}^T C^{-1}\underline{\gamma}\}$$

where s stands for the variance s_x or s_y of the additional noise and $\underline{\gamma}$, C stand for $\underline{\gamma}^{\theta,k}$ and $C^{\theta,k}$ respectively.

Taking logarithms,

$$-2\ln p(\underline{\gamma}^{j,k}|H_1,\theta,s) = -2\ln\pi(j,\theta-1) + n\ln 2\pi + \ln|C| + \underline{\gamma}^T C^{-1}\underline{\gamma}$$

Let,

$$f(\theta,s) \triangleq -2\ln\pi(j,\theta-1) + \ln|C| + \underline{\gamma}^T C^{-1}\underline{\gamma} \qquad \text{(IV.20)}$$

The minimum of $f(\theta,s)$ with respect to θ and s has to be found.

C can be written as the sum of two terms as:

$$C = \text{diag}[c(i,i)] + sC'$$

$$= E + sC'$$

where C' is given by (III.77) or (III.91).

Unfortunately, in this case the maximisation of $f(\theta,s)$ does not yield compact analytical solutions. Consider for example,

$$\frac{\partial f(\theta,s)}{\partial s} = \text{tr}\left[C^{-1}\frac{dC}{ds}\right] - \underline{\gamma}^T C^{-1}\frac{dC}{ds}C^{-1}\underline{\gamma}$$

$$= \text{tr}\left[C^{-1}C'\right] - \underline{\gamma}^T C^{-1}C'C^{-1}\underline{\gamma} \qquad \text{(IV.21)}$$

At a minimum (IV.21) vanishes. Hence to find $\hat{s}$ (IV.21) has to be solved for every $\theta\varepsilon\Theta$ and the pair (θ,s) that minimise (IV.20) chosen.

The variance s may be bounded, but an added requirement is that

$$C > 0 \quad \text{or} \quad \underline{x}^T(E+sC')\underline{x} > 0; \quad \text{all } \underline{x}.$$

Since the entries in E are variances, E is positive definite. Let,

$$\underline{x} = E^{-\frac{1}{2}}W\underline{z}$$

where W is the orthogonal matrix of e-vectors of $E^{-\frac{1}{2}}C'E^{-\frac{1}{2}}$ and $W^{-1} = W^T$. Then,

$$\underline{z}^T W^T E^{-\frac{1}{2}} E E^{-\frac{1}{2}} W\underline{z} + s\underline{z}^T W^T E^{-\frac{1}{2}} C' E^{-\frac{1}{2}} W\underline{z} > 0; \quad \text{all } \underline{z}$$

or $$\underline{z}^T\underline{z} + s\underline{z}^T\Lambda\underline{z} > 0,$$

where Λ is the diagonal matrix of real e-values of $E^{-\frac{1}{2}}C'E^{-\frac{1}{2}}$. Then

$$\underline{z}^T(I+s\Lambda)\underline{z} > 0; \quad \text{all } \underline{z} \quad \text{or}$$

$$1+s\lambda_i > 0 ; \quad \text{all } i$$

If the e-values λ_i are all positive, then the condition

$$s > 0$$

is sufficient. If negative e-values exist,

$$0 < s < -\frac{1}{\bar{\lambda}_{min}}$$

where $\bar{\lambda}_{min}$ is the smallest negative e-value.

The numerical minimisation of the likelihood equation, given by (IV.20), may be performed in the case of scalar systems by one of several methods.

Firstly, in situations where the minimum s is known to lie in a finite interval, numerical minimisation algorithms for functions of one variable can be used. Such a method, using cubic polynomial curve fitting is used in the simulation tests.

Secondly, as it can be seen from (IV.21), $\frac{\partial f(\theta,s)}{\partial s}$ is a polynomial

of degree 2n-1, whose coefficients can be calculated (Appendix I.4) and its real roots extracted by numerical methods. An exhaustive search may then be carried out to determine the minimising root.

Thirdly, if the interval in which s is constrained is sufficiently small, it may be discretized. The value of s that minimises (IV.20), can then be found by searching through the interval. This procedure would have to be repeated for every θ, and the minimum pair (θ, s) picked.

The number of subdivisions will depend on the required accuracy, though for most applications a hundred steps should be adequate.

The above methods will work satisfactorily for scalar dynamical systems, where s will be a real positive scalar. The amount of computation required should not prove prohibitive for modern computer systems. A different approach must however be adopted in multivariable systems, though it should be pointed out that the difficulties in obtaining the absolute maximum of the likelihood function should not invalidate the proposed method. In such cases the values of the likelihood function at different points in the parameter space may be used to make relative statements about the likelihood of one parameter value versus the another. Also,in cases where the observation vector can be processed as a sequence of scalar components, scalar minimisation procedures may be used.

The evaluation of (IV.20) requires the computation of the determinant and the inverse of C. This can be done either by direct computational methods or by orthogonalising the residual sequence. The orthogonalisation results in a diagonal covariance matrix whose inverse and determinant can be found trivially. The Gram-Schmidt iterative procedure can be used to sequentially uncorrelate the residuals. The following recurrence relations are used *[57]*:

$$c_{ww}(\theta,\theta) = c(\theta,\theta)$$

$$c_w(i,j) = c(i,j) - \sum_{m=\theta}^{j-1} \frac{c_w(i,m)\,c_w(j,m)}{c_{ww}(m,m)}$$

$$c_{ww}(i,i) = c(i,i) - \sum_{m=\theta}^{i-1} \frac{c_w^2(i,m)}{c_{ww}(m,m)}$$

$$\gamma_w(i) = \gamma(i) - \sum_{m=\theta}^{i-1} \frac{c_w(i,m)\,\gamma_w(m)}{c_{ww}(m,m)} \qquad \text{(IV.22)}$$

where $\gamma_w(i)$ is the whitened residual sequence,

$$c_w(i,j) = \text{cov}[\gamma_w(i),\gamma(j)]$$

$$c_{ww}(i,j) = \text{cov}[\gamma_w(i),\gamma_w(j)]$$

and $c(i,j) = \text{cov}[\gamma(i),\gamma(j)]$ as given by one of (III.77), (III.91).

Using the new whitened sequence, the determinant and inverse of its diagonal covariance matrix is given by:

$$|C_{ww}| = \prod_{i=\theta}^{k} c_{ww}(i,i)$$

and $$\ln|C_{ww}| = \sum_{i=\theta}^{k} \ln c_{ww}(i,i)$$

Also $$C_{ww}^{-1} = \begin{bmatrix} c_{ww}^{-1}(\theta,\theta) & & \\ & \ddots & \\ & & c_{ww}^{-1}(k,k) \end{bmatrix}$$

Hence,

$$\underline{\gamma}_w^T\, C_{ww}^{-1}\, \underline{\gamma}_w = \sum_{i=\theta}^{k} \frac{\gamma_w^2(i)}{c_{ww}(i,i)}$$

Having calculated $(\hat{\theta},\hat{s})$ by some numerical method, their values are substituted into the likelihood function and the GLR statistic obtained

as:

$$2\ln\Lambda_g = 2\ln p(\underline{\gamma}^{j,k}|H_1,\hat{\theta},\hat{s}) - 2\ln p(\underline{\gamma}^{j,k}|H_0) \underset{H_0}{\overset{H_1}{\gtrless}} 2\ln\lambda$$

$$= -\ln|C^{\hat{\theta},k}| - [\underline{\gamma}^{\hat{\theta},k}]^T[C^{\hat{\theta},k}]^{-1}\underline{\gamma}^{\hat{\theta},k} +$$

$$+ \sum_{i=\hat{\theta}}^{k}\{\ln c(i,i) + \frac{\gamma^2(i)}{c(i,i)}\} \underset{H_0}{\overset{H_1}{\gtrless}} 2\ln\lambda \qquad \text{(IV.23)}$$

IV.1.3.5 Determination of P_f, P_d, λ and n for the GLR tests in cases (a) or (d).

The computational complexity and consequent computing time for the implementation of the GLR method depends largely on the residual sequence sample size n.

The equations linking P_f, P_d, λ and n are *[44]*:

$$P_f = \int_{\lambda}^{\infty} p(\Lambda_g|H_0)\,d\Lambda_g \qquad \text{(IV.24)}$$

$$P_d(\theta,\nu) = \int_{\lambda}^{\infty} p(\Lambda_g|H_1,\theta,\nu)\,d\Lambda_g \qquad \text{(IV.25)}$$

To calculate P_f, P_d, λ and n, the following procedure may be adopted: calculate λ by fixing P_f and then choose n to give a reasonable tradeoff between P_d and detection delay time t_d. However (IV.25) depends on both θ and ν and its evaluation has to be performed for some arbitrary pair. A sensible choice is to fix θ and ν at the minimum value that needs to be detected.

The logarithm of the GLR statistic in cases (a) or (d) is given by (IV.19). It is shown in *[58]*, that $p(\ln\Lambda_g|H_0)$ is a χ^2 density with one degree of freedom, while $p(\ln\Lambda_g|H_1,\theta,\nu)$ is a non-central χ^2 density with non-centrality parameter,

$$\delta^2 = \nu^2 \sum_{m=\theta}^{k} \frac{a^2(m)}{c(m,m)} \qquad \text{(IV.26)}$$

Therefore, having fixed P_f, λ can be calculated using the values of the χ^2 distribution with one degree of freedom. These are given in Table 8.

TABLE 8

Percentage points of the χ^2 distribution (P_f) *[51]*.

P_f	0.5	0.25	0.1	0.05	0.025	0.01	0.005
$\ln\lambda$	0.455	1.32	2.7	3.84	5.02	6.63	7.88

The calculation of P_d depends on the value of the non-centrality parameter δ^2 which, as mentioned earlier, depends on θ, ν and n. Table 9 gives values of P_d for different P_f and δ^2.

TABLE 9

Power of the non-central χ^2 distribution (P_d) *[59]*

δ^2 \ P_f	0.01	0.005
5	.146	.284
10	.449	.638
15	.712	.856
20	.881	.952
25	.956	.985
30	.985	.996

The entries in Table 9 show that increasing δ^2 results in an improved P_d. It can also be seen from (IV.26) that for fixed θ and ν the value of the parameter depends on $k-\theta$. Since the terms in the summation are positive, increasing the sample size increases δ^2. Thus P_d increases, but, as discussed in Section IV.1.2.1.1(a), detection delay time increases and a tradeoff study has to be performed for specific applications.

IV.1.3.6 Determination of P_f, P_d, λ and n for GLR tests in cases (c) or (f).

In the case of additional plant or measurement noise the form of the GLR statistic, given by (IV.23), does not permit any analytical results, since neither of $p(\ln\Lambda_g|H_o)$, $p(\ln\Lambda_g|H_1,\theta,s)$ can be obtained analytically.

The threshold value, λ, has to be chosen by experiment. A computer simulation method for the determination of P_d, P_f and λ in cases where the pdf of the likelihood function under the hypotheses cannot be written in analytical form, is given in *[60]*. The method makes use of the fact, that,

$$p(\ln\Lambda_g|H_1) = \ln\Lambda_g p(\ln\Lambda_g|H_o)$$

Thus P_d may be written equivalently as

$$P_d = \int_{\tilde{\lambda}}^{\infty} \tilde{\Lambda}_g p(\tilde{\Lambda}_g|H_o)\, d\tilde{\Lambda}_g \quad ; \quad \tilde{\Lambda} = \ln\Lambda_g$$

Therefore, the density under H_o actually need be obtained. This can be estimated from Monte Carlo simulation of the monitored process in no-fault conditions. Let such an estimate be:

$$\hat{p}(\tilde{\Lambda}_g|H_o)$$

Then the relevant equations involving P_d, P_f and $\tilde{\lambda}$ can be written:

$$\hat{P}_f = \int_{\tilde{\lambda}}^{\infty} \hat{p}(\tilde{\Lambda}_g | H_0) d\tilde{\Lambda}_g \qquad \text{(IV.27)}$$

$$\hat{P}_d = 1 - \int_{0}^{\tilde{\lambda}} \tilde{\Lambda}_g \hat{p}(\tilde{\Lambda}_g | H_0) d\tilde{\Lambda}_g \qquad \text{(IV.28)}$$

Equation (IV.28) is suitable from the standpoint of utilising numerical simulation results *[60]*.

IV.1.3.7 Simplifications of the GLR algorithm.

The implementation of the GLR algorithm involves the calculation of $\tilde{\Lambda}_g$ at every time k. This is quite straightforward in cases (a) or (d), but becomes complicated in cases (c) or (e), where (IV.20) has to be maximised, not just evaluated, for every value of θ in the sampling interval. As a result, if the values over which θ is maximised are reduced, then the amount of necessary computation will also be less.

A simplified GLR algorithm will therefore result, if

$$j < \theta < k-m$$

where m is an integer whose value may be determined by considering the tradeoff between P_d and t_d. This scheme will detect faults with a delay of at least m time units.

A further simplification will result if no optimisation over θ is performed. This will mean that θ is fixed to some value, say k-m, and optimisation is then performed to identify the size of the fault. The value m should be chosen large enough for the estimation to be performed accurately.

If the system and filter have reached steady state, the steady state values of the mean and correlation of residuals following a fault can be considered.

The steady state values of the mean in cases (a), (d) are:

$$\frac{\eta\nu_x}{1-s} \quad \text{and} \quad \left\{\frac{\eta\phi K}{s-1}+1\right\}\nu_y \quad \text{respectively , where } s=(1-K\eta)\phi$$

while the covariance in steady state following a fault in cases (c), (f), is:

$$\text{cov}[\gamma(i),\gamma(j)] = \delta_{i,j}c + s^{i-j}\left\{\frac{\eta^2 s_x}{1-s^2}\right\}$$

$$\text{cov}[\gamma(i),\gamma(j)] = -s^{i-j}\left\{\frac{(\eta\phi K)^2}{s^2-1}+\eta\phi K s^{-1}\right\}s_y \quad ; \quad i\neq j$$

$$= c - \left\{\frac{(\eta\phi K)^2}{s^2-1} - 1\right\}s_y \quad ; \quad i=j$$

A GLR algorithm based on steady state behaviour following a fault, should work quite satisfactorily, if the time taken to reach steady state is sufficiently small. The identification properties of the test will not be affected as far as the fault size is concerned. The time of the fault occurrence cannot be identified directly but can be estimated from the knowledge of the time that the system takes to settle after the fault occurrence.

The two hypotheses in cases (a), (d) may be written:

$$H_0: \gamma(k) = \gamma_0(k)$$
$$H_1: \gamma(k) = \gamma_0(k) + b_i\nu$$

where b_i denotes the corresponding constant in (III.64) or (III.81) Since $a(m)=b_i$=constant, $c(m,m)=c$, (IV.16) becomes:

$$\hat{\nu} = \frac{\sum_{m=j}^{k} \gamma(m)}{nb_i}$$

Hence the GLR statistic corresponding to (IV.19) is:

$$\ln\Lambda_g = \frac{\{\sum_{m=j}^{k} \gamma(m)\}^2}{nc} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\lambda$$

Similar simplifications will result in the case of additional noise in the plant or measurements.

Finally, if the filter has reached steady state, i.e. K(k)=K, equations (III.22)-(III.24) and (III.47)-(III.50) describing the evolution of the various g functions in cases (a), (c), (d) and (f) depend only on (i-j) and will only have to be calculated once.

IV.1.3.8 Implementation of GLR algorithms.

Regardless of whether the GLR algorithm is used in conjunction with the simpler detection-partial isolation algorithms or not a rule for termination of the algorithm must be defined.

Since the accuracy in the estimation of a parameter increases with increasing sample size, a reasonable procedure would be to terminate the GLR algorithm when,

$$\tilde{\Lambda}_g > \tilde{\lambda}$$

and

$$\hat{\theta} = j$$

Such a choice means that the algorithm will terminate when the estimated time of fault occurrence coincides with the first residual in the batch and the GLR statistic exceeds the calculated threshold. Therefore information about the fault will exist in the maximum number

of residuals.

If optimisation over θ is not performed, but a value $\theta=k-m$ assumed, then the GLR algorithm may terminate when,

$$\tilde{\Lambda}_g > \tilde{\lambda}$$

in which case $\hat{\theta}=k-m$.

In all cases reinitialisation of the GLR parameters is necessary, following the detection-estimation of a fault. In this way, successive faults can be detected, provided, as mentioned in Section II.4.1, that the faults occur sufficiently apart for the fault monitoring to reorganise the system.

IV.2 Distinguishability between faults of the same class.

The successful operation of any fault monitoring scheme depends on its ability to distinguish between faults which have similar effects on the observed variable, in particular on the filter residuals. Specifically, the following questions must be answered:

1. Is it possible that the effect of a step bias in the state is "seen" by the fault monitoring scheme as the effect of a step bias in the measurements and conversely ?

2. Is it possible that the effect of additional plant noise is "seen" by the fault monitoring scheme as the effect of additional measurements noise and conversely ?

Consider the effect of a bias. This is given by (III.25) or (III.46) as,

$$\sum_{i=\theta}^{k} g_a(k,i)\nu_x \qquad \text{and} \qquad \sum_{i=\theta}^{k} g_d(k,i)\nu_y$$

in the cases of state and measurement bias respectively. As shown in Appendix I.1, equivalent expressions are:

$$\sum_{i=\theta}^{k} \eta s^{k-i}\nu_x = \eta\frac{1-s^{k-\theta+1}}{1-s}\nu_x$$

and

$$\sum_{i=\theta}^{k} \{-\eta\phi K s^{k-1-i} + 1\}\nu_y = \{1 + \eta\phi K\frac{1-s^{k-\theta}}{s-1}\}\nu_y$$

If the two events are indistinguishable, then there exist ν_x, ν_y such that,

$$\eta\,\frac{1-s^{k-\theta+1}}{1-s}\,\nu_x = \{1 + \eta\phi K\,\frac{1-s^{k-\theta}}{s-1}\}\nu_y \quad , \quad \text{all } k$$

It is easy to see that this cannot happen. Consider $\theta=k$, then $\nu_x = \eta^{-1}\nu_y$. If $k=\theta+1$,

$$\eta\,\frac{1-s^2}{1-s}\nu_x = \{1+\eta\phi K\,\frac{1-s}{s-1}\}\nu_y$$

or, $\eta(1+s)\nu_x = (1-\eta\phi K)\nu_y$

Substituting for ν_x gives:

$$\eta(1+s)\eta^{-1}\nu_y = (1-\eta\phi K)\nu_y$$

or, $1 + (1-\eta K)\phi = 1 - \eta\phi K$

or, $\phi = 0$, contradiction.

Additional noise affects the residual covariance. The following expressions, developed in Appendix I.1, can be used to infer

distinguishability:

$$\mathrm{cov}[\gamma(k),\gamma(m)] = \eta^2 s^{k+m-2\theta}\frac{1-s^{-2(k-\theta+1)}}{1-s^{-2}}\, s_x$$

and

$$\mathrm{cov}[\gamma(k),\gamma(m)] = \left\{(\eta\phi K)^2\,\frac{s^{k+m-2\theta}-s^{m-k}}{s^2-1} - \eta\phi K s^{k-m-1}\right\} s_y$$

If the effects are indistinguishable there would exist s_x and s_y such that the above expressions are equal for all k, m. Consider $k+m=2\theta$, then

$$\eta^2\,\frac{s^2-s}{s^2-1}\,s_x = \left\{(\eta\phi K)^2\,\frac{1-s}{s^2-1} - \eta\phi K\right\} s_y$$

After calculations,

$$s_x = \frac{-\phi K\{\frac{\eta\phi K}{s+1}+1\}}{\eta\frac{s}{1+s}}\, s_y$$

Letting $K+m=2\theta+1$, and substituting for s_x, yields,

$$\frac{-\eta^2 s\phi K\{\frac{\eta\phi K}{s+1}+1\}}{\frac{s}{1+s}} = -\eta\phi K s\{\frac{\eta\phi K}{s+1}+1\}$$

or,

$$1+s = s \;;\ \text{contradiction}$$

These results show that the proposed schemes are able to distinguish between similar faults.

where $\hat{q}$, $\hat{r}$ are calculated using the fact that the variance of the sum of two independent random variables is the sum of their variances,

$$\hat{q} = q + s_x$$
$$\hat{r} = r + s_y$$

The operation of updating the state estimate and the state estimate error variance, can be carried out as follows: equations (III.11), (III.13) describe the state and state estimate for the system subject to faults, as:

$$x_{tr}(k) = x_o(k) + h_x(k,\theta,\Delta p) \qquad \text{(III.11)}$$

$$\hat{x}_{fil}(k/k) = \hat{x}_o(k/k) + f(k,\theta,\Delta p) \qquad \text{(III.13)}$$

where the subscripts tr and fil have been used to emphasize that the corresponding parameters are the true and the output of the employed Kalman filter respectively.

It can be seen from (III.11) that the optimal state estimate is, after taking expectations:

$$E[\hat{x}_{tr}(k)\,|\,y^k] = \hat{x}_o(k/k) + E[h_x(k,\theta,\Delta p)\,|\,y^k]$$
$$= \hat{x}_{fil}(k/k) - f(k,\theta,\Delta p) + E[h_x(k,\theta,\Delta p)\,|\,y^k] \qquad \text{(IV.31)}$$

Thus, in the case of bias in the state or measurements the optimal state estimate will be, after substitution:

$$\hat{x}_{tr}(k/k) = \hat{x}_{fil}(k/k) - \sum_{i=\theta}^{k} \{f_a(k,i)\nu_x + \phi^{k-i}\nu_x\}$$

or

$$= \hat{x}_{fil}(k/k) - \sum_{i=\theta}^{k} \{f_d(k,i)\nu_y + \phi^{k-i}\nu_y\}$$

respectively, where f_a and f_d are given by (III.23) and (III.48). In both cases the correction involves nonrandom quantities whose values can be found exactly. This fact does not neccessitate an update in the error variance. This procedure is also intuitively correct, since the value of

CHAPTER V: *SIMULATION RESULTS*

V.1 Computer based simulation.

The proposed methods for fault monitoring were tested by Monte Carlo simulation runs. The simulated system is:

$$x(k+1) = 0.7\, x(k) + w(k)$$

$$y(k) = x(k) + v(k)$$

with $E[w(k)]=0$, $E[v(k)]=0$, all k, $\bar{x}(0)=0$, and,

$E[w^2(k)]=0.3$, $E[v^2(k)]=0.3$, $p(0)=0.5$

The steady state Kalman filter parameters for this system, labelled TS4 in the test runs, are: K=0.562, c=0.68, p=0.17. Table 13 is a computer output of the matrices $C_c^{\theta,k}$ and $C_c^{\theta,k}$ defined by (III.17), (III.91) respectively, while Table 14 shows the residual bias sequence given by (III.63), (III.80) in the cases of state and measurement bias respectively.

To facilitate the possibility of introducing a change in ϕ of a larger magnitude without causing instabilities tests were also carried out with $\phi=0.3$. This system is referred to as TS3.

All computer programs were written in Fortran and structured in such a way so that maximum usage of data files created by individual programs was made. In this way computation time and storage requirements were reduced. Results were plotted on a visual display unit and subsequently hardcopied. Numerical Algorithm Group routines were used for the numerical maximisation of the likelihood function and for the generation of pseudo-random numbers. These two routines are described in Appendix II.2. All computer runs were carried out on CDC-6400 Time-sharing computer system.

V.2 Description of tests.

The tests were divided into two parts. In the first, the validity and performance of the simple detection-partial isolation tests were examined, while in the second part the performance of the GLR tests was investigated.

The main Kalman filter program was directed to operate on a sequence of one hundred noisy measurements and therefore generated one hundred filter residuals. A fault of any kind and size could be applied to the system at any time and the corresponding residual sequence stored to a file for subsequent analysis by the fault monitoring program.

In the first part, tests were carried out for every fault condition. In the second part, tests were restricted to types (a), (c), (d), (f) in view of the increase in problem and programming complexity.

V.2.1 Tests for detection-partial isolation algorithms.

Plotted results for these tests are shown in figs. 12-20. Every figure is a collection of four graphs with a common heading describing the state of the simulated system and the number, n, of residuals in the batch. The four graphs show:

1. Top LHS: variation of residuals
2. Top RHS: variation of number of positive residuals as used in the sign test for the mean.
3. Bottom LHS: variation of residual variance.
4. Bottom RHS: variation of first order serial correlation.

The residual sequence is plotted against time-step k, while the values of the statistics are calculated at each time step k, using a batch of

n residuals. Therefore the first plotted value does not appear until k=n+m. The delay m is the time taken for the filter variance to settle. Within .0001, simulations show that for TS4, m=4.

If the values of the UCL and LCL lines are within the plot boundaries, they are plotted together with their values. The length of the axes and the coordinate values shown are chosen automatically by the plotting routine.

Three tests involving a change in the state transition coefficient were performed. A fault which did not destabilise the system as well as a destabilising one were introduced in TS4, while TS3 was used to test the effects of a change of size 0.6.

The performance of the tests under different fault conditions is summarised in Table 10. If the calculated statistics confirmed their predicted behaviour a 'Yes' entry is indicated, otherwise a 'No' appears. In the case of a bias, for example, the number of positive residuals should exceed its limits while the variance and first order serial correlation should remain within their bounds. This would indicate that the residuals have nonzero mean but are independent a condition which can only arise if a bias exists in either the state or the measurements. If, on the other hand, additional noise is simulated, the reverse should happen, indicating that the residuals have nonzero mean but are correlated, a condition that arises if additional noise exists.

The entry in first parenthesis for (b) denotes results obtained using TS4 with destabilising fault, while the second parenthesis denotes results obtained with TS3.

Table 11 shows the detection delay time, t_d, given as the number of time steps, for each kind of fault (where inapplicable a - is shown).

TABLE 10

	sign test	variance	r_1
No fault	yes	yes	yes
a. State bias	yes	yes	no
b. change in ϕ.	yes (yes) (no)	no (yes) (no)	no (yes) (yes)
c. additional plant noise	yes	yes	yes
d. measurement bias	yes	yes	yes
e. change in η.	yes	yes	no
f. additional measurement noise	yes	yes	no

TABLE 11

	sign test	variance	r_1
a.	5	-	-
b.	-(-) (16)	-(-) (26)	38 (9) (9)
c.	-	10	9
d.	22	-	-
e.	-	10	-
f.	-	23	-

TABLE 12

	UCL	LCL	P_f
sign test	19	11	0.1
	22	8	0.01
sample variance	1.03	0.43	0.05
	1.16	0.33	0.01
first order serial correlation	0.27	-0.33	0.1
	0.327	-0.387	0.05

The residual sequence length was chosen so that the probability of false alarms, P_f, was low. Since in a real application, faults will occur infrequently it is important to keep the probability of false alarms low, though there may be situations where a high false alarm rate could be tolerated in return for high probability of detection.

The quantities UCL, LCL and P_f, for n=30, for the various tests are given in Table 12. These are calculated using the methods and Tables of Sections IV.1.2.1.1 and IV.1.2.1.2.

The value of P_d depends on the actual value of the statistic used. Tables 5 and 6 can be used to find P_d for the sign test. In the case of the variance test, P_d can be calculated using the steady state values of the variance of the residuals as follows:

(c). Additional plant noise: the true variance is given by (III.78) for i=j as,

$$\sigma_1^2 = \frac{\eta^2}{1-s^2} s_x + c = \frac{1}{1-(1-0.568)^2 0.7^2} + 0.682 = 1.787$$

The variable λ^2 used in (IV.10) is therefore:

$$\lambda^2 = \frac{1.787}{0.682} = 2.62$$

and $$\frac{\chi^2_{29,0.025}}{2.62} = 6.12$$

$$\frac{\chi^2_{29,0.975}}{2.62} = 17.45$$

Hence, $$P_d = P[\chi^2_{29}<6.12] + p[\chi^2_{29}>17.45] \simeq 0.95$$

This value of P_d is not very high because λ^2, which indicates the relative increase in variance, is quite low. As can be seen from fig. 10, high values of P_d can be obtained for $\lambda^2>4$.

(f). Additional measurement noise: the true variance is given by (III.89) as:

$$\sigma_1^2 = c + \left\{1 - \frac{(\eta\phi K)^2}{s^2-1}\right\} s_y = 1.852$$

Hence $\lambda^2 = 2.715$ and

$$\frac{\chi^2_{29,0.025}}{2.715} = 5.91, \quad \frac{\chi^2_{29,0.975}}{2.715} = 16.83$$

and P_d is calculated from (IV.10) $\simeq 0.961$.

Fig. 21 shows a pair of tests for the mean using the sample mean statistic and a preliminary pair of tests for correlation using the rank correlation coefficient.

As it can be seen from the bottom pair of graphs the rank test of independence did not perform at all well in two tested cases of additional plant and measurement noise and was thus eliminated from the possible tests of independence. The reason for the failure is not clear.

The top pair of graphs verifies the results of Section IV.1.2.1.1. The top LHS graph shows a reduction of the sample mean variance, while the top RHS shows an increase in the sample mean variance.

Using (III.78) the correlation between $\gamma(j)$ and $\gamma(i)$ is calculated as,

$$\text{cov}[\gamma(i),\gamma(j)] = \frac{(0.3024)^{i-j}}{0.9}$$

in the case of additional plant noise, while in the case of additional measurement noise (III.88) yields,

$$\text{cov}[\gamma(i),\gamma(j)] = -0.2276(0.3024)^{i-j}$$

These results predict that the additional factor required to adjust the limits of the sample mean, as given by (IV.6), will be positive in the case of additional plant noise, hence an increase in the limits is required, while it will be negative in the case of additional measurement noise

hence a decrease in the limits is required. This is clearly shown in fig. 21.

V.2.1.1 Conclusions for detection-partial isolation algorithms.

The summarised results of Table 10, together with the inapplicability of the mean and rank correlation tests suggest the procedure shown in fig. 22. This procedure could correctly detect every fault in the simulated system and would not give any false alarms.

By comparison, the variance test seems to be much better than the first order serial correlation test, with only one missed detection, that in the case of faults in ϕ, of size$\leq$0.2, 0.6. This might be due to the small fault size, which would produce small λ^2 and consequently small P_d. With this in mind, the variance test could be used if faults that produce appreciable effects only need be detected.

The detection delay times, shown in Table 11 can be improved in certain cases if the sample size is reduced. However, this will increase the probability, P_f, of false alarms and a final choice must be made with the requirements of a particular application in mind.

V.2.2 GLR tests.

The GLR tests are shown in figs. 23-29 in groups of 3. The top graph shows the variation of the GLR statistic $\tilde{\Lambda}_g$, the middle graph the MLE of the size of the fault and the bottom graph shows the MLE of the time of fault occurrence. All parameters are plotted against time step k. The graph heading shows the state of the system, the length of the residual string, n, and the number m of the optimisation window for θ as described in Section IV.1.3.7. "Simple" GLR refers to the case where the null hypothesis is hypothesis of no fault. This

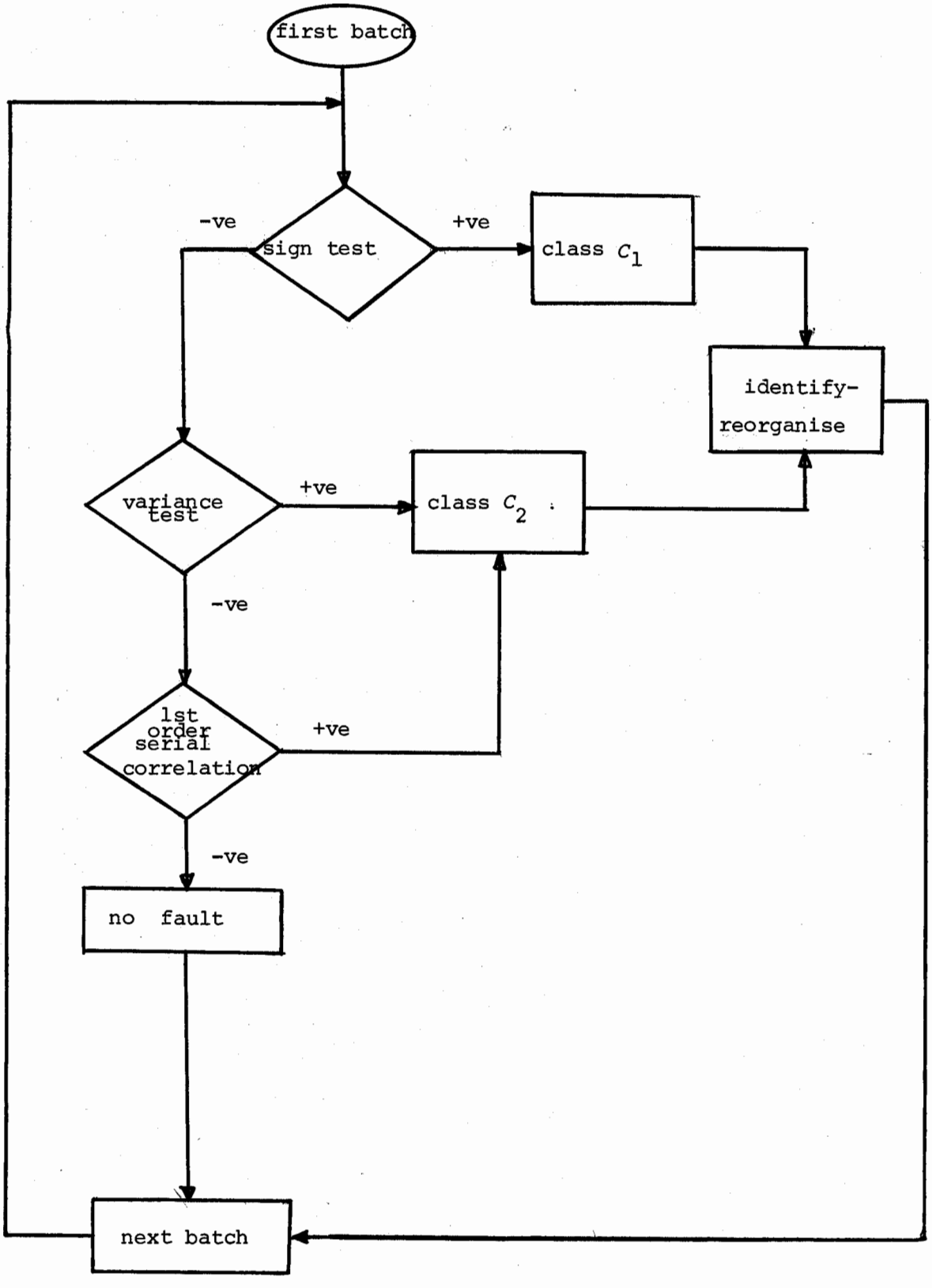


fig.22 Detection-partial isolation procedure

would be the case if the GLR algorithm is used on its own.

The algorithm, and plotting, stops when a fault has been detected and completely identified, i.e. when

$$\tilde{\Lambda}_g > \tilde{\lambda} \qquad \underline{\text{and}}$$
$$\theta = j$$

Various simulation runs were performed with faults applied to the means and variances of the noise sequences. In all cases the faults were correctly detected, isolated and estimated.

Figs. 23-25 show the effect of increasing the sample size and window length, m, on the time and size estimates. It may be seen that increasing the sample size results in substantially better estimates of size, while time estimates are relatively unaffected.

Tables 8 and 9 can be used to calculate P_f and P_d relative to λ and n. Hence if $\lambda=8$, $P_f=0.005$. To calculate P_d, in the case of bias in the state or measurements, the non-centrality parameter δ^2, appearing in (IV.26) must be calculated:

$$\delta^2 = \nu_x^2 \sum_{m=\theta}^{k} \frac{a^2(m)}{c(m,m)}$$

where $a(m) = \sum_{i=\theta}^{m} g_a(m,i)$. In the case of a step bias, it is shown in Appendix I.1(a) that, $g_a(m)=\eta s^m$; $s=(1-K\eta)\phi$. Hence,

$$a(m) = \eta \sum_{i=0}^{m-\theta} s^i = \eta \frac{1-s^{m-\theta+1}}{1-s}$$

and if $c(m,m)=c$,

$$\delta^2 = \frac{1}{c} \sum_{m=\theta}^{k} \left\{\eta \frac{1-s^{m-\theta+1}}{1-s}\right\}^2 \nu_x^2 > 30 \text{ if } k-\theta=18$$

Hence, from table 9, $P_d>0.996$.

Figures 26-27 show corresponding tests in the case of step bias in the measurements. A larger fault size was needed in this case, since the increase in the value of the residual mean is relatively lower by a factor of 3 in this case than in the case of state bias. This can be seen by evaluating the steady state value of the mean in the two cases using equations (III.64) and (III.81). For TS4 the values are 1.44 and 0.43 for the state and measurement bias respectively.

Table 15 shows results for a "multiple" GLR test. The results are in tabulated rather than plotted form because the differences are small and would not be obvious in a graph. The simulated system is subject to a state step bias with the shown parameters. The LHS part of the table shows the GLR algorithm results obtained using the correct GLR detector for this case, i.e. the state bias detector. The RHS of the table shows results for the same system and using the same n, m, but employing a GLR detector designed to monitor measurement bias. The results show that the GLR values obtained using the correct detector are consistently higher than the ones obtained via the incorrect detector. This implies that a procedure based on choosing the fault with the highest GLR value, would consistently make correct identification of the occurred fault in cases where more than one kind of fault is likely to occur, though not simultaneously.

Figs. 28, 29 show simulation runs of the system with additional state and measurement noise respectively. The feature of these tests is the large sample size needed for the accurate estimation of the fault size. Preliminary tests with smaller sample sizes produced unacceptable results. The maximisation of the fault size was constrained from 0 to 10.

With the chosen sample size, the fault size and time of fault occurrence were correctly identified.

Table 16 shows results for a multiple GLR test in the case of additional noise in the measurements. The outcome is the same as the one described for the multiple GLR test in the case of a bias. The pair of these tests imply that if the GLR procedure is used in conjunction with the simple detection-partial isolation statistical techniques, described by the flowchart of fig. 22, the resulting monitoring scheme would correctly detect and identify faults (a), (c), (d) and (f), any of which is likely to occur at any time in a dynamical system described by (I.1)-(I.2).

V.2.2.1 Conclusions for the GLR tests.

The validity of the GLR algorithm was succesfully tested in the cases of "simple" and "multiple" hypotheses.

Thus, in cases where the monitoring of only one kind of fault is required, the results would indicate that GLR procedures work well and have smaller detection delay times than simple detection-partial isolation algorithms based on sample statistics. Correct identification of the size and time of fault requires a longer delay. Thus in applications where a rapid detection of faults is required, GLR tests should be preferred.

The GLR detectors perform equally well in cases of where more than one type of fault is likely to occur. In such cases, if used in conjunction with the simpler tests, the resulting fault monitoring process will accurately detect and estimate any faults in the form of bias in the state or measurements or in the form of additional plant or measurement noise.

TABLE 13

CORRELATION MATRIX IN THE EVENT OF INCREASED MEASUREMENT NOISE

1.000	-.392	-.121	-.037	-.011	-.004	-.001	-.000
-.392	1.154	-.345	-.106	-.033	-.010	-.003	-.001
-.121	-.345	1.168	-.340	-.105	-.032	-.010	-.003
-.037	-.106	-.340	1.170	-.340	-.105	-.032	-.010
-.011	-.033	-.105	-.340	1.170	-.340	-.105	-.032
-.004	-.010	-.032	-.105	-.340	1.170	-.340	-.105
-.001	-.003	-.010	-.032	-.105	-.340	1.170	-.340
-.000	-.001	-.003	-.010	-.032	-.105	-.340	1.170

CORRELATION MATRIX IN THE EVENT OF INCREASED PLANT NOISE

1.000	.308	.095	.029	.009	.003	.001	.000
.308	1.095	.337	.104	.032	.010	.003	.001
.095	.337	1.104	.340	.105	.032	.010	.003
.029	.104	.340	1.105	.340	.105	.032	.010
.009	.032	.105	.340	1.105	.340	.105	.032
.003	.010	.032	.105	.340	1.105	.340	.105
.001	.003	.010	.032	.105	.340	1.105	.340
.000	.001	.003	.010	.032	.105	.340	1.105

TABLE 14

RESIDUAL BIAS IN THE EVENT OF BIAS IN THE MEASUREMENTS

1.00000	.607748	.487033	.449884	.438451
.434932	.433850	.433516	.433414	.433382
.433373	.433370	.433369	.433368	.433368

RESIDUAL BIAS IN THE EVENT OF BIAS IN THE STATE

1.00000	1.30775	1.40246	1.43160	1.44057
1.44333	1.44418	1.44444	1.44452	1.44455
1.44456	1.44456	1.44456	1.44456	1.44456

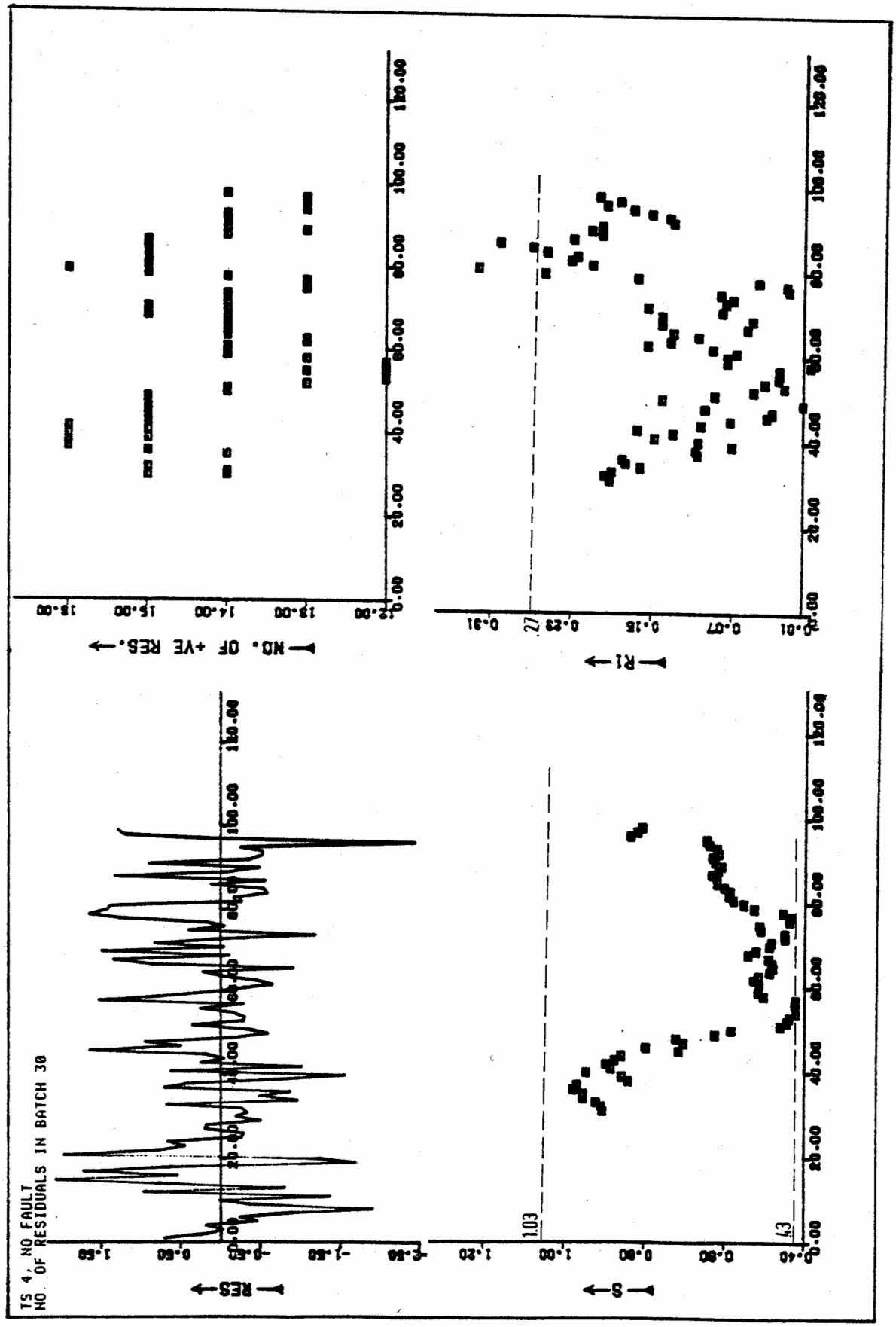

TS 4, NO FAULT
NO OF RESIDUALS IN BATCH 30
RES
S
R1
NO. OF +VE RES.
1.03
.43
.27

Fig. 12

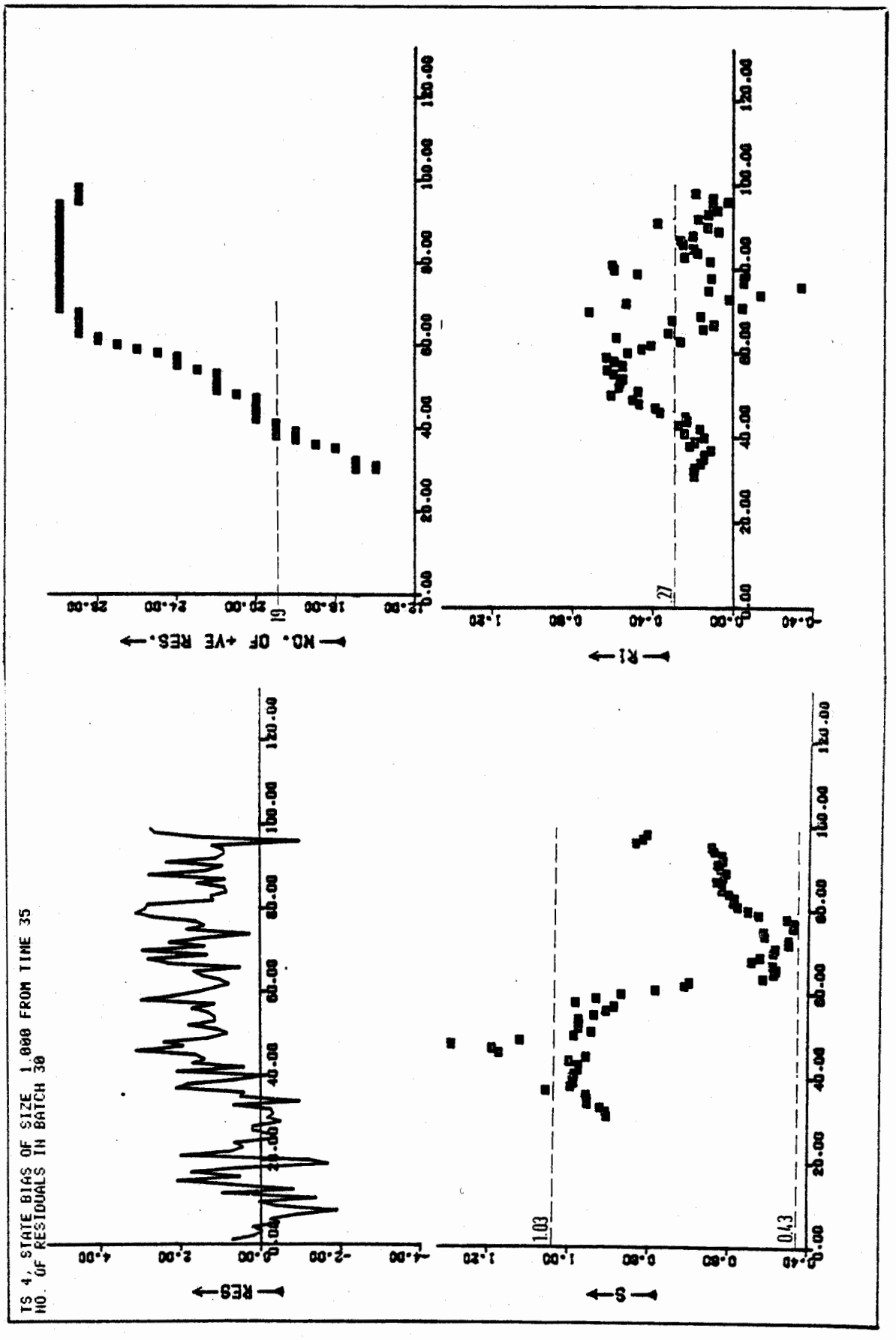

TS 4, STATE BIAS OF SIZE 1.000 FROM TIME 35
NO. OF RESIDUALS IN BATCH 30
RES
S
1.03
.043
R1
27
NO. OF +VE RES.
61

Fig. 13

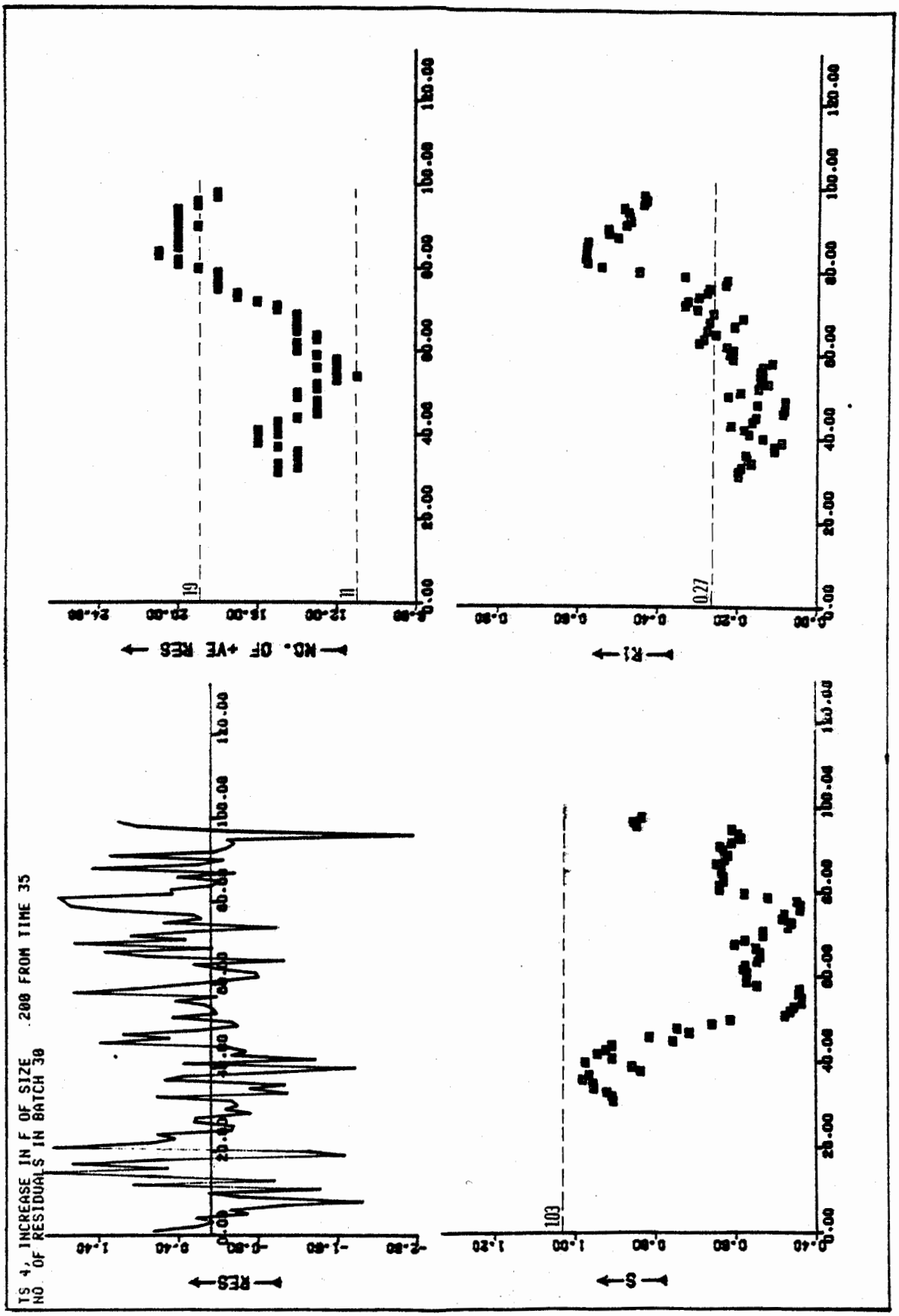

TS 4, INCREASE IN F OF SIZE .200 FROM TIME 35
NO OF RESIDUALS IN BATCH 30
RES
S
R1
NO. OF +VE RES
0.27
103
19
11

Fig. 14

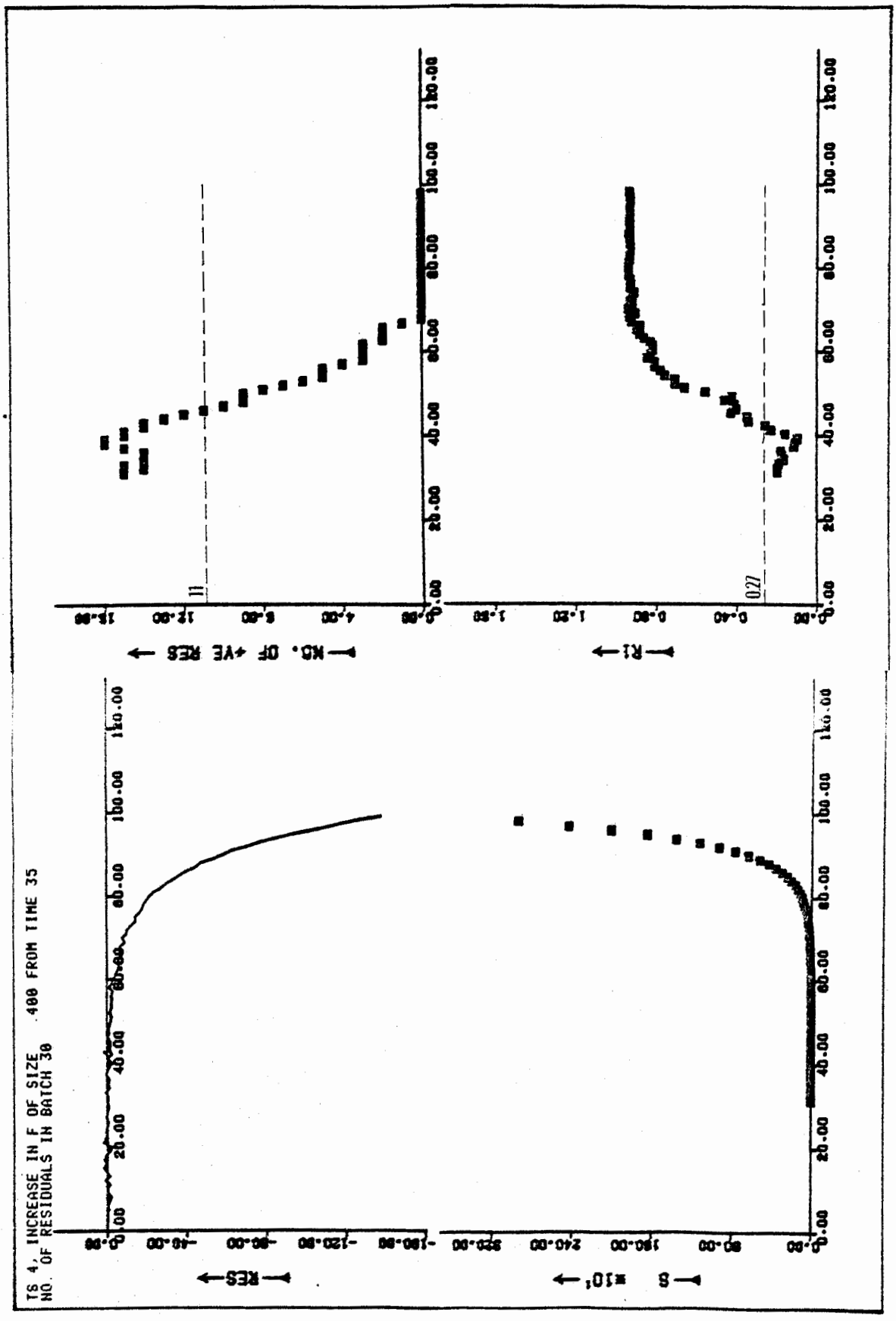

TS 4, INCREASE IN F OF SIZE .400 FROM TIME 35
NO. OF RESIDUALS IN BATCH 30
RES
S ×10³
R1
NO. OF +VE RES
11
0.27

Fig. 15

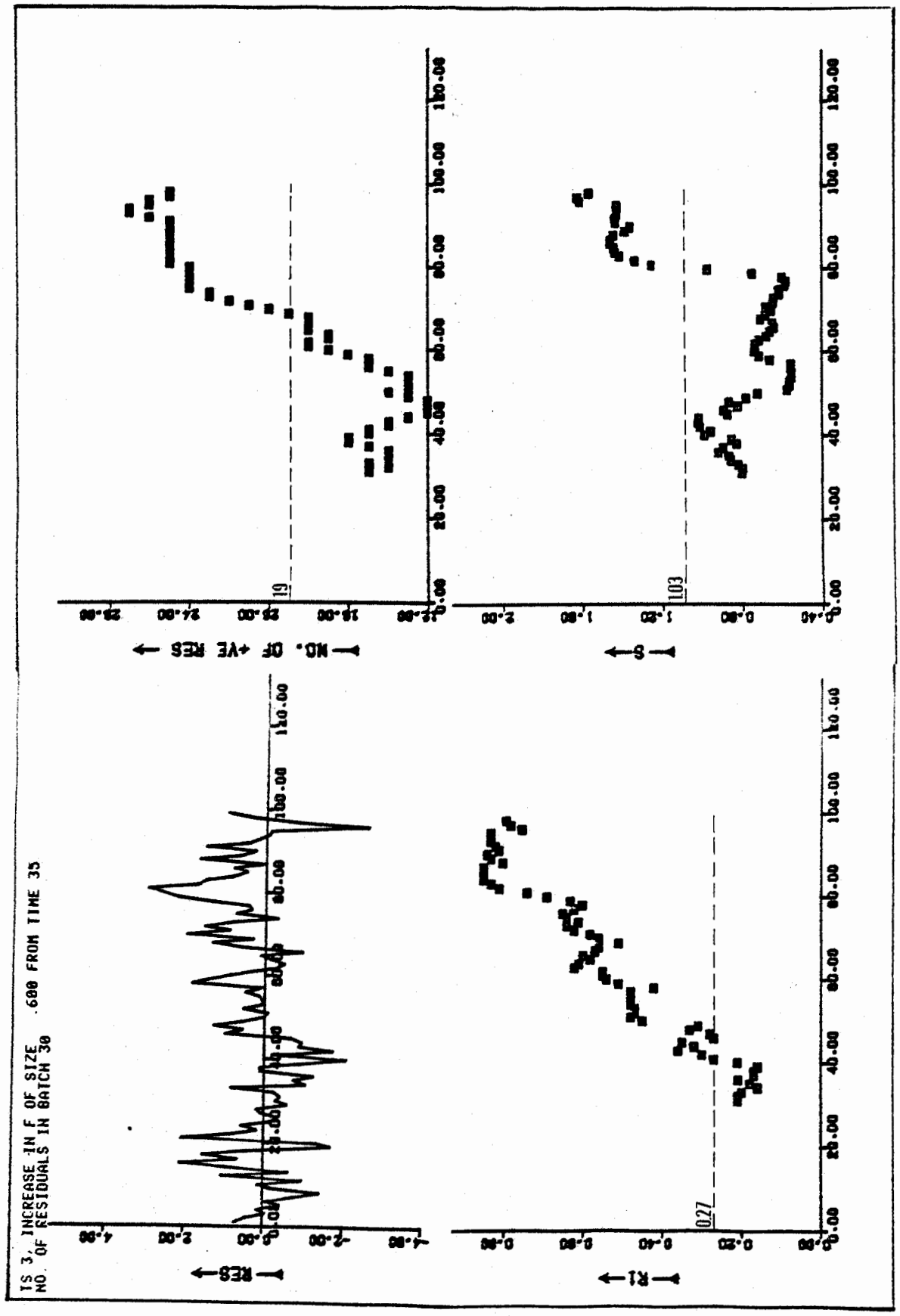

IS 3, INCREASE IN F OF SIZE .600 FROM TIME 35
NO. OF RESIDUALS IN BATCH 30
NO. OF +VE RES
RES
S
R1
19
103
027

Fig. 16

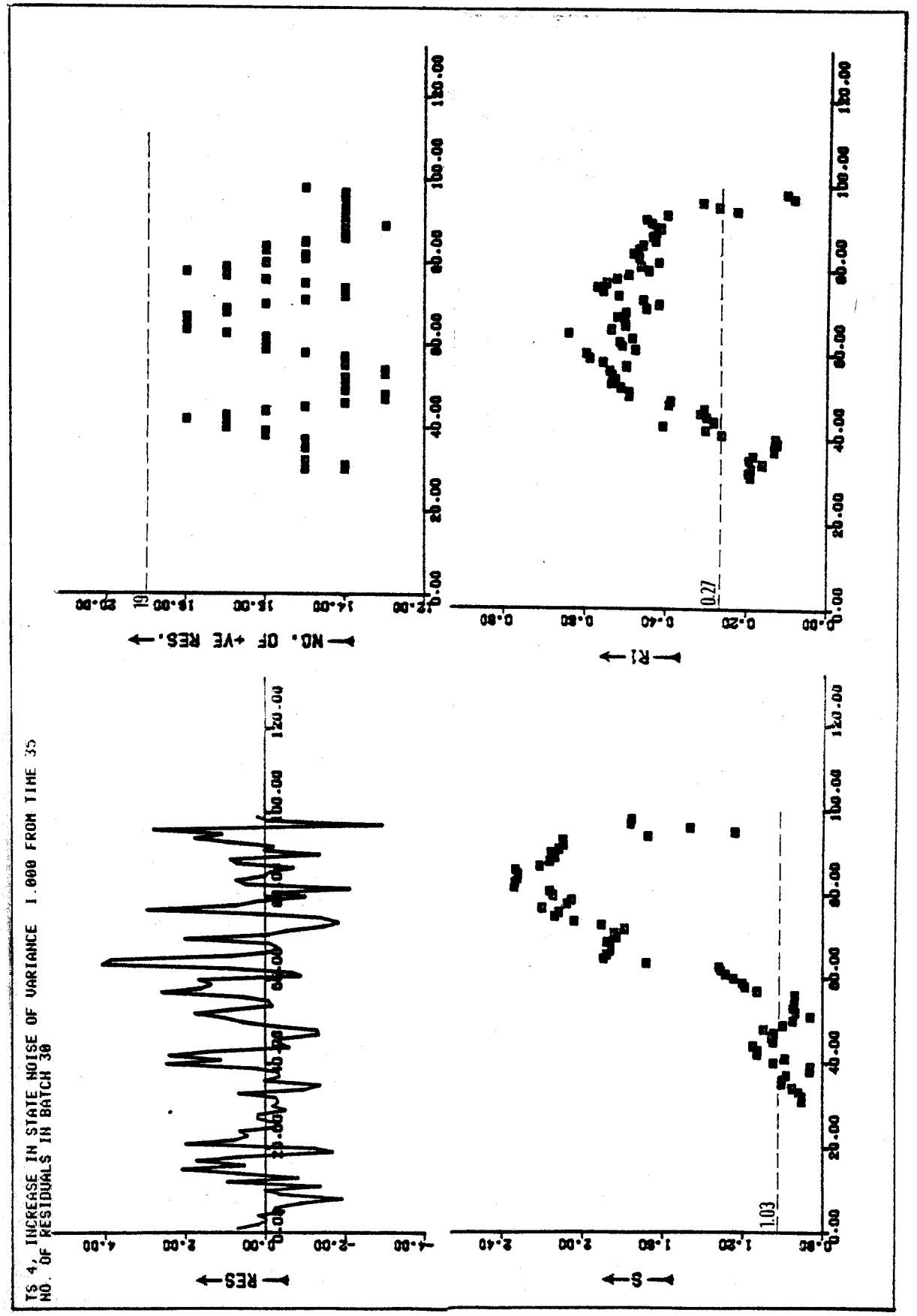

TS 4, INCREASE IN STATE NOISE OF VARIANCE 1.000 FROM TIME 35
NO. OF RESIDUALS IN BATCH 30
RES
S
R1
NO. OF +VE RES.
0.27
1.03
19

Fig. 17

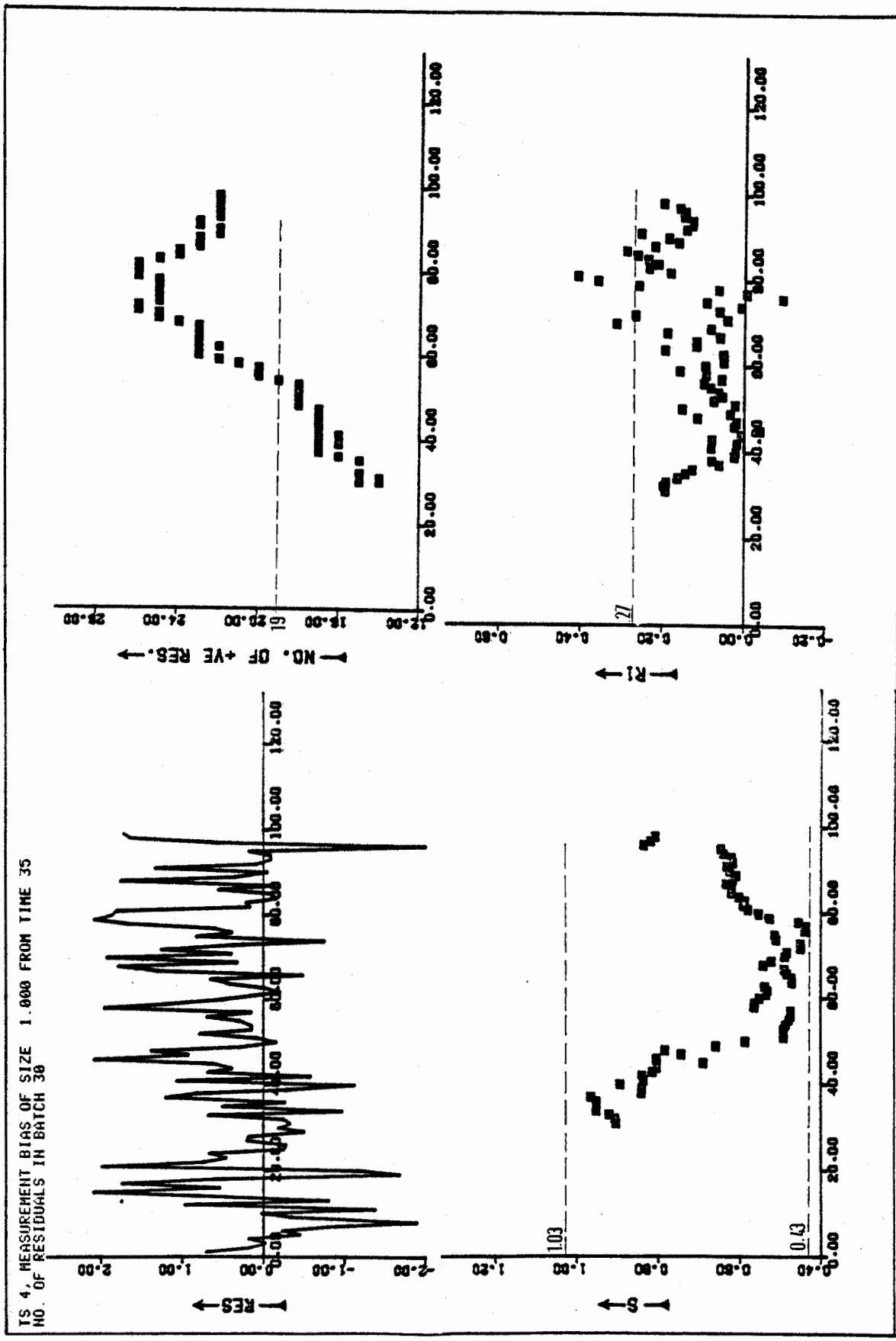

TS 4, MEASUREMENT BIAS OF SIZE 1.000 FROM TIME 35
NO. OF RESIDUALS IN BATCH 30
RES
S
R1
NO. OF +VE RES.
1.03
0.43
27
19

Fig. 18

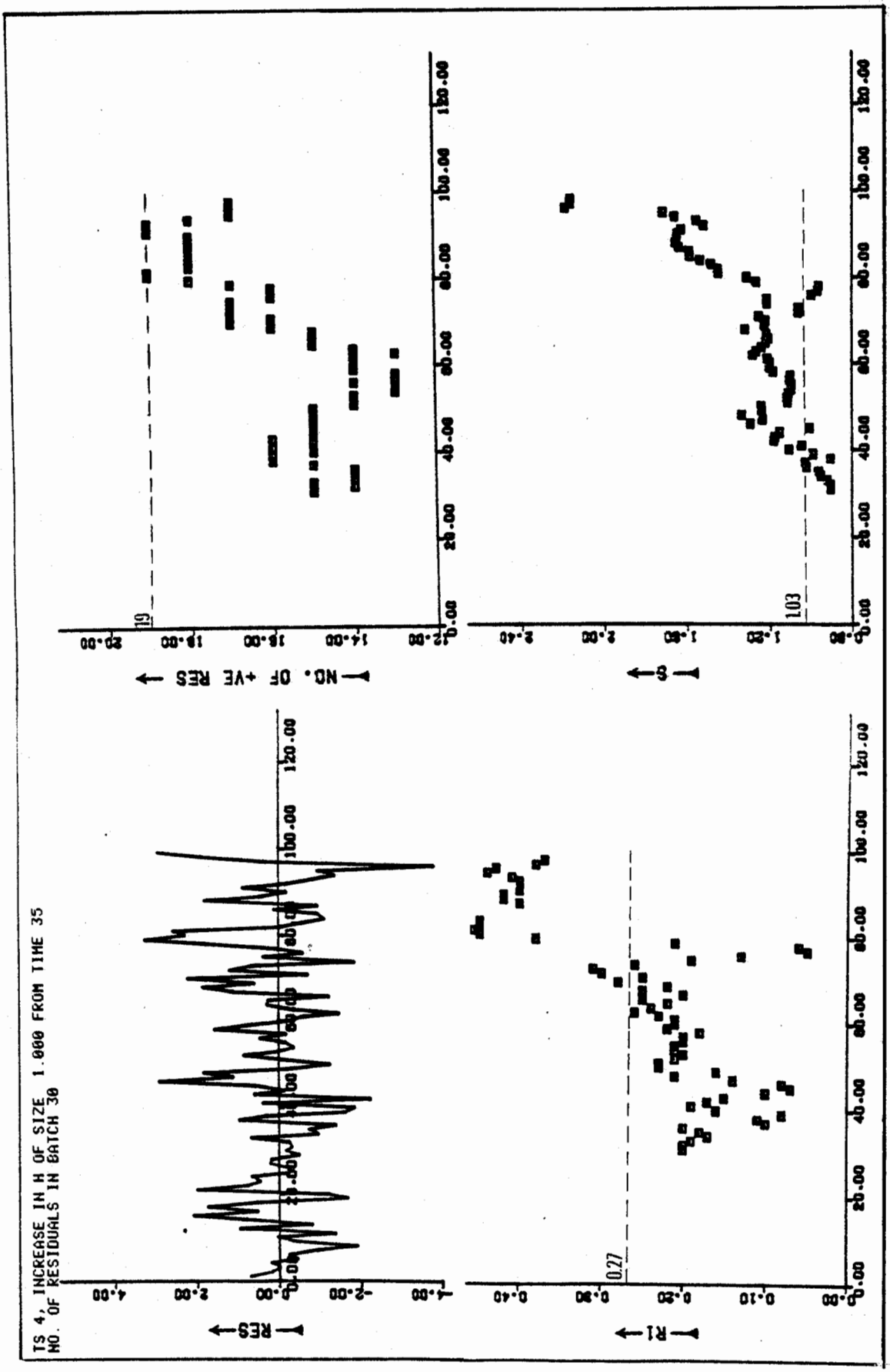

TS 4, INCREASE IN H OF SIZE 1.000 FROM TIME 35
NO. OF RESIDUALS IN BATCH 30
RES
RI
S
NO. OF +VE RES
0.27
103
19

Fig. 19

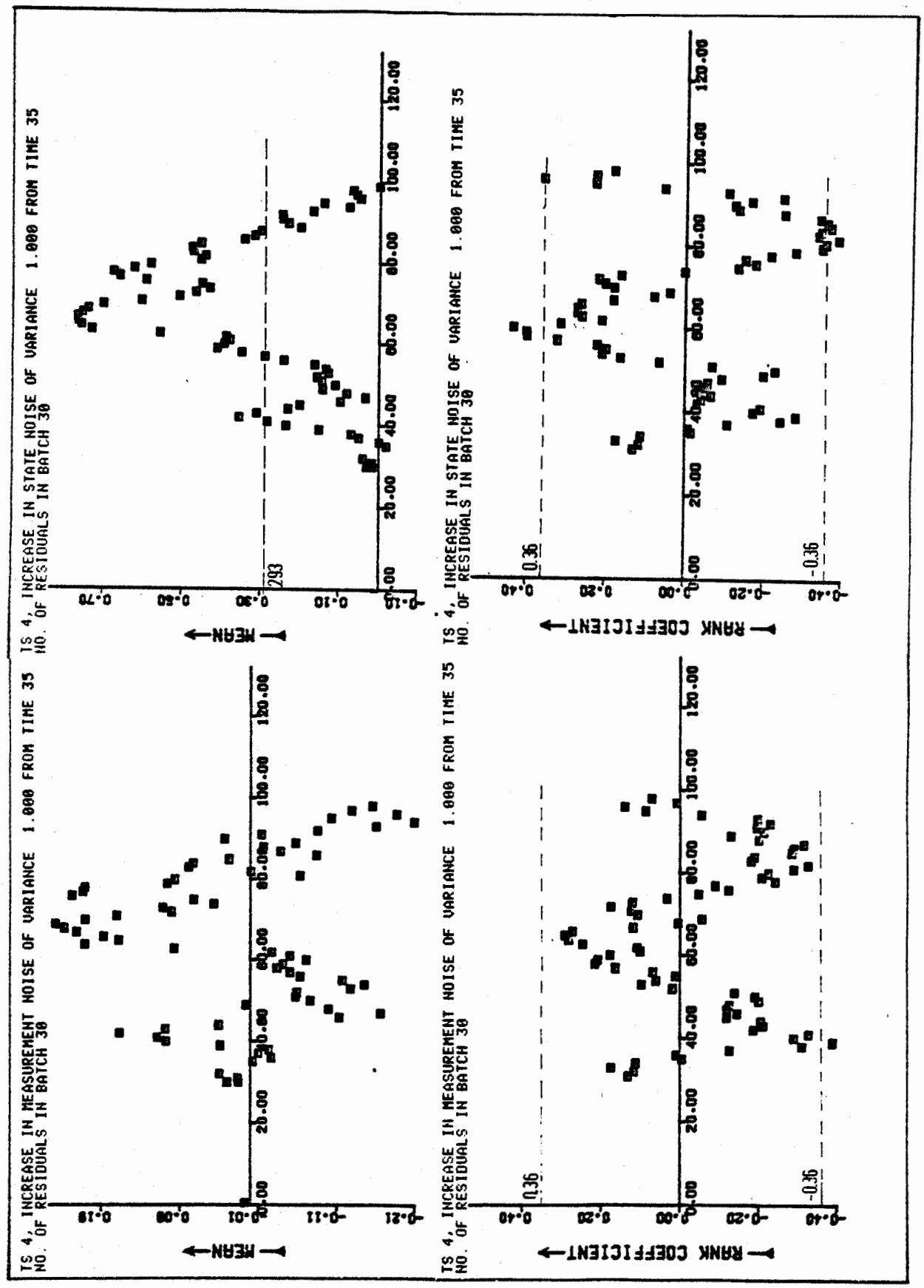

TS 4, INCREASE IN STATE NOISE OF VARIANCE 1.000 FROM TIME 35
NO. OF RESIDUALS IN BATCH 30
MEAN
293
RANK COEFFICIENT
0.36
-0.36
TS 4, INCREASE IN MEASUREMENT NOISE OF VARIANCE 1.000 FROM TIME 35
NO. OF RESIDUALS IN BATCH 30
MEAN
RANK COEFFICIENT
0.36
-0.36

Fig. 21

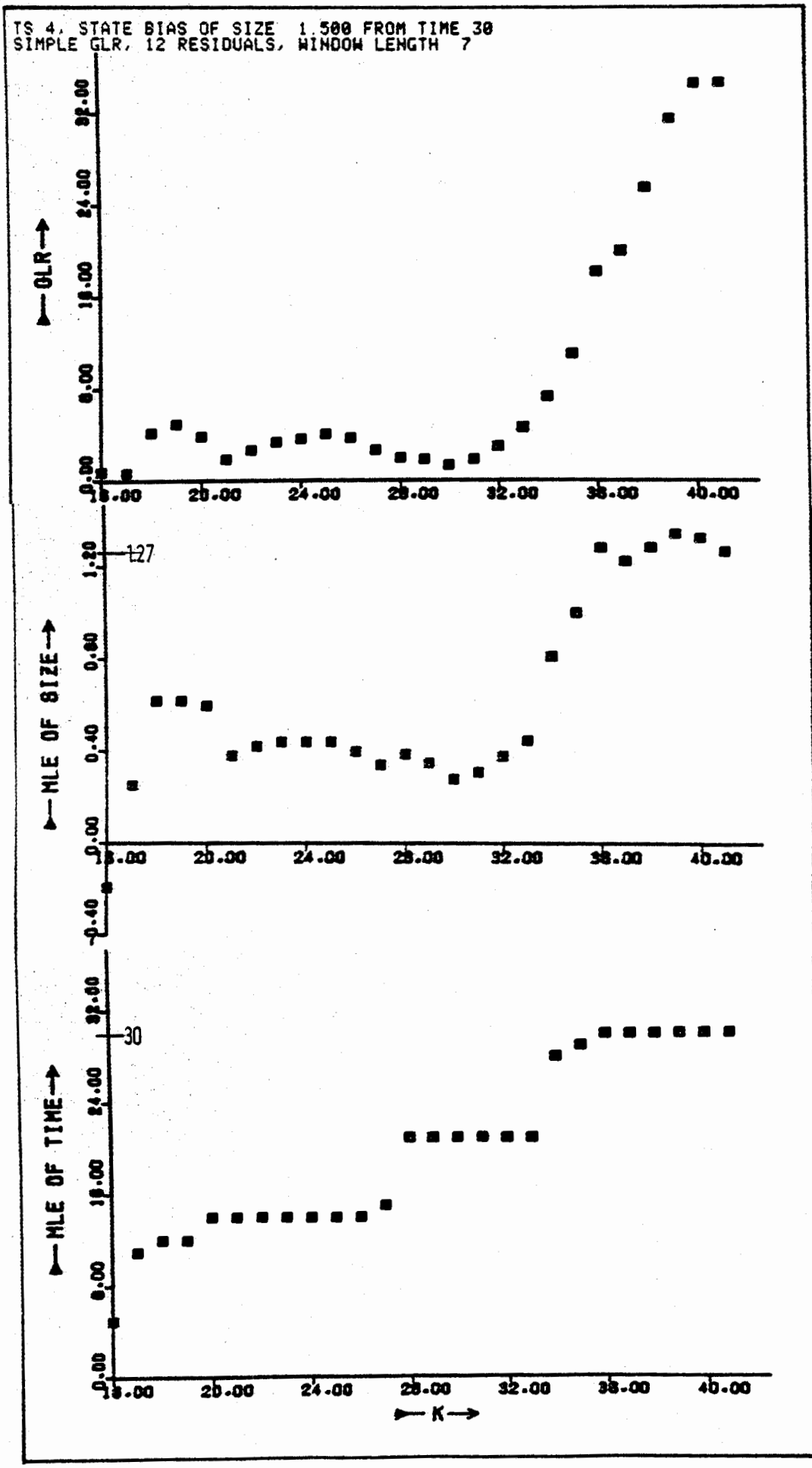
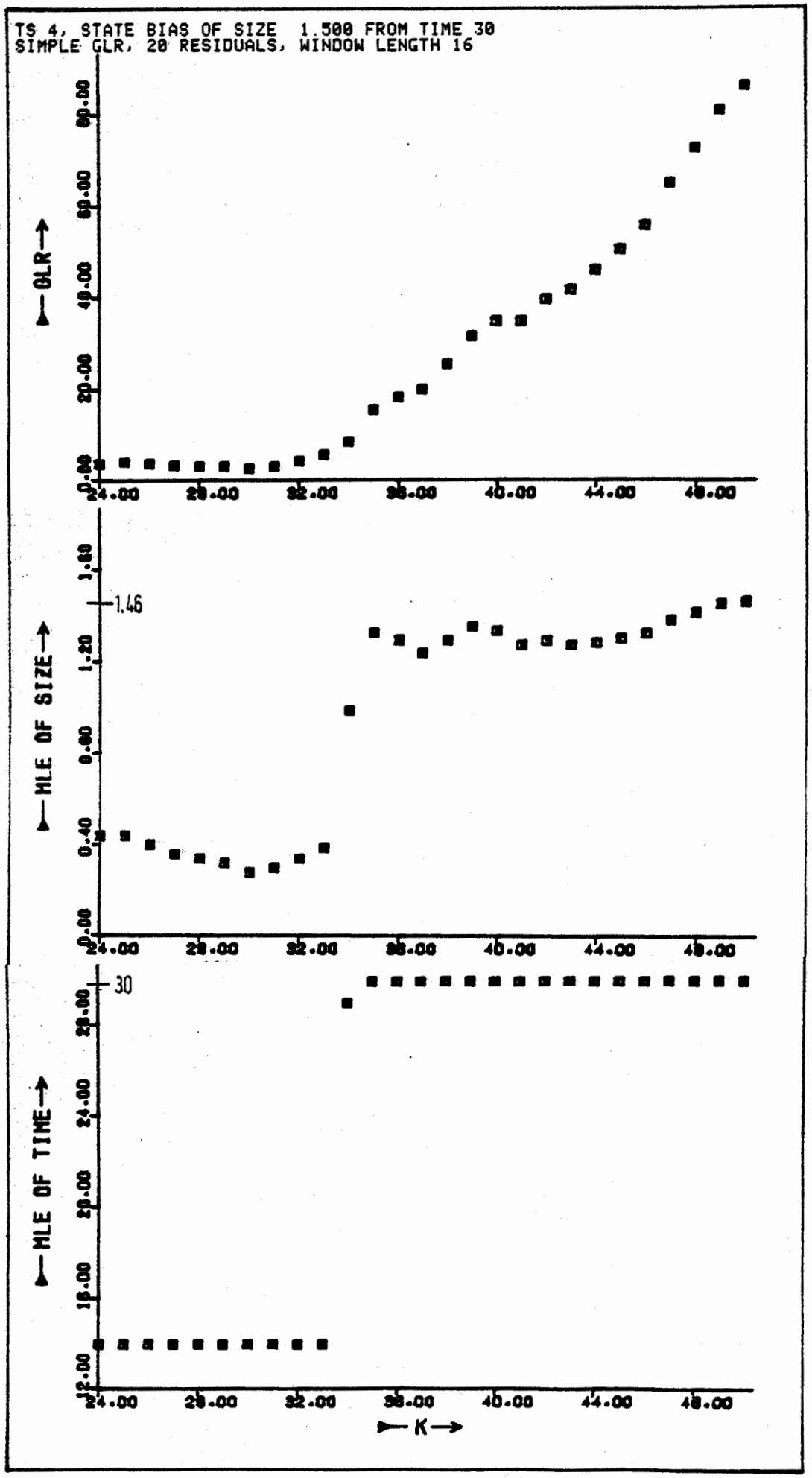

TS 4, STATE BIAS OF SIZE 1.500 FROM TIME 30
SIMPLE GLR, 12 RESIDUALS, WINDOW LENGTH 7
GLR
MLE OF SIZE
MLE OF TIME
K
127
30

Fig.23

fig. 24


TS 4, STATE BIAS OF SIZE 1.500 FROM TIME 30
SIMPLE GLR, 20 RESIDUALS, WINDOW LENGTH 16
GLR
MLE OF SIZE
1.46
MLE OF TIME
30
K
24.00
28.00
32.00
36.00
40.00
44.00
48.00

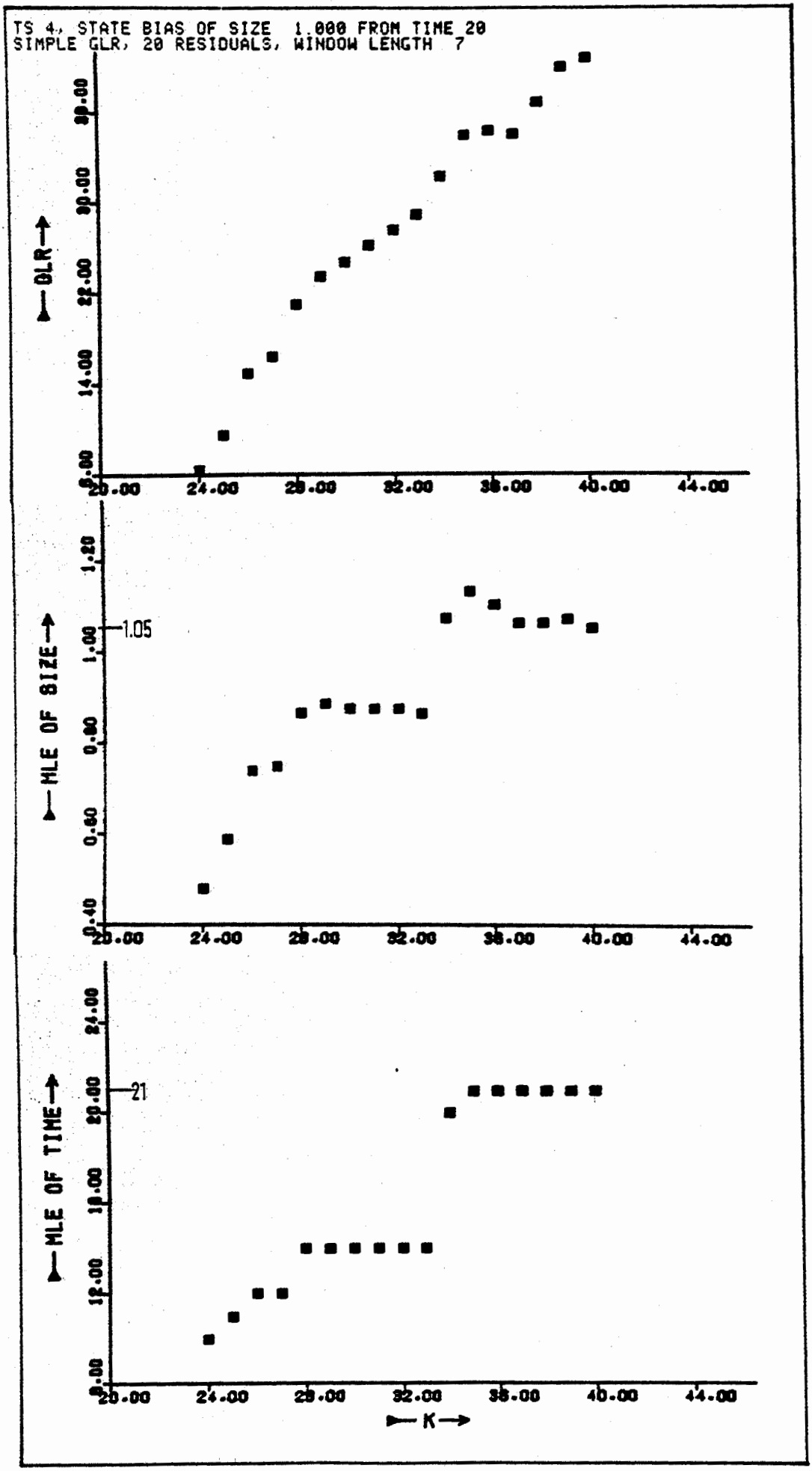

TS 4, STATE BIAS OF SIZE 1.000 FROM TIME 20
SIMPLE GLR, 20 RESIDUALS, WINDOW LENGTH 7
GLR
MLE OF SIZE
1.05
MLE OF TIME
21
K
20.00
24.00
28.00
32.00
36.00
40.00
44.00

Fig. 25

fig.26

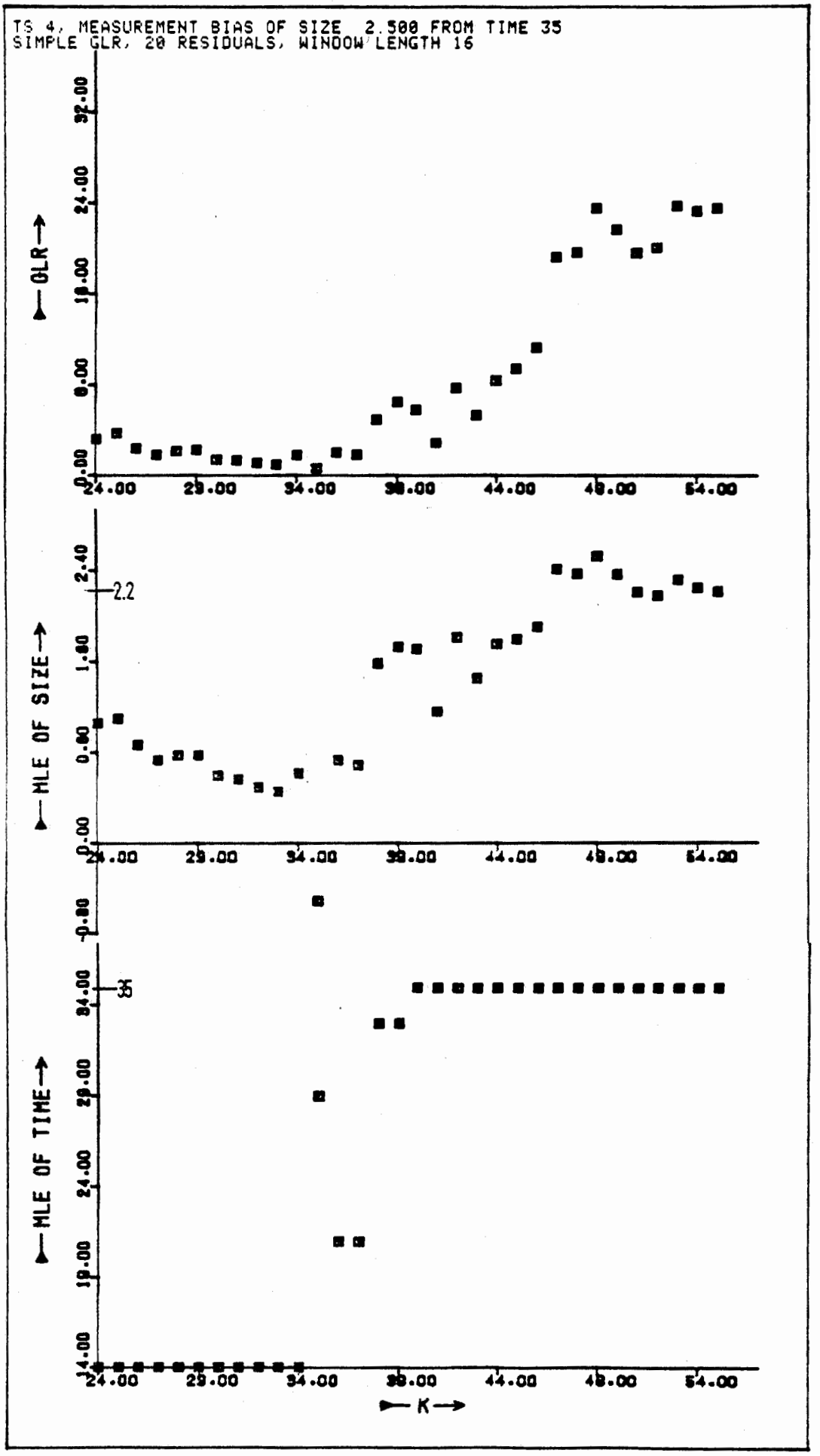

TS 4, MEASUREMENT BIAS OF SIZE 2.500 FROM TIME 35
SIMPLE GLR, 20 RESIDUALS, WINDOW LENGTH 16
GLR
MLE OF SIZE
2.2
MLE OF TIME
35
K
24.00
29.00
34.00
39.00
44.00
49.00
54.00

fig.27

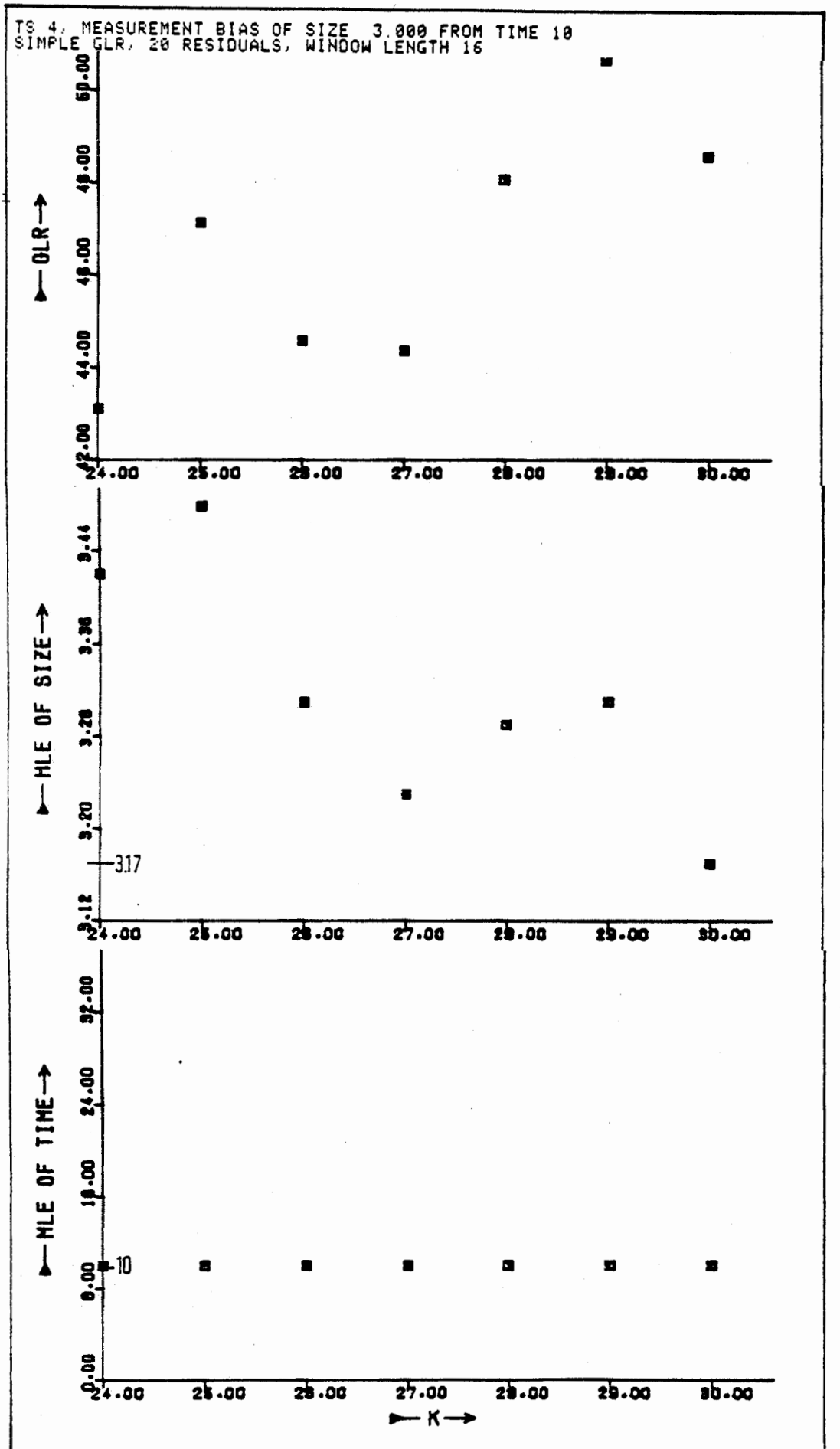

TS 4, MEASUREMENT BIAS OF SIZE 3.000 FROM TIME 10
SIMPLE GLR, 20 RESIDUALS, WINDOW LENGTH 16
GLR
MLE OF SIZE
3.17
MLE OF TIME
10
K
24.00
25.00
26.00
27.00
28.00
29.00
30.00

TABLE 15

TEST SYSTEM 4, STATE BIAS OF SIZE 1.5 FROM TIME 30

SIMPLE GLR, 20 RESIDUALS, WINDOW LENGTH 16

GLR	SIZE	TIME
3.71,	.44,	14
4.12,	.44,	14
3.77,	.40,	14
3.27,	.36,	14
3.10,	.34,	14
3.02,	.32,	14
2.58,	.28,	14
3.09,	.30,	14
4.16,	.34,	14
5.67,	.39,	14
8.60,	.99,	29
15.60,	1.33,	30
18.27,	1.30,	30
20.05,	1.24,	30
25.59,	1.30,	30
31.62,	1.36,	30
34.75,	1.34,	30
34.85,	1.28,	30
39.70,	1.30,	30
41.76,	1.28,	30
46.23,	1.29,	30
50.69,	1.31,	30
55.78,	1.33,	30
65.07,	1.39,	30
72.61,	1.42,	30
80.95,	1.46,	30

MULTIPLE GLR, 20 RESIDUALS, WINDOW LENGTH 16

GLR	SIZE	TIME
3.30,	1.06,	14
3.82,	1.10,	14
2.43,	.86,	14
1.85,	.73,	14
2.21,	.77,	14
2.32,	.77,	14
1.43,	.59,	14
3.57,	.92,	14
5.77,	1.14,	14
8.11,	1.32,	14
13.09,	2.56,	29
13.14,	2.56,	30
15.88,	2.69,	30
16.45,	2.63,	30
27.10,	3.25,	30
33.06,	3.46,	30
31.02,	3.25,	30
26.81,	2.93,	30
39.20,	3.44,	30
36.36,	3.22,	30
44.82,	3.48,	30
48.95,	3.55,	30
54.85,	3.67,	30
61.64,	3.70,	30
70.76,	3.98,	30
80.93,	4.02,	30

Fig. 28

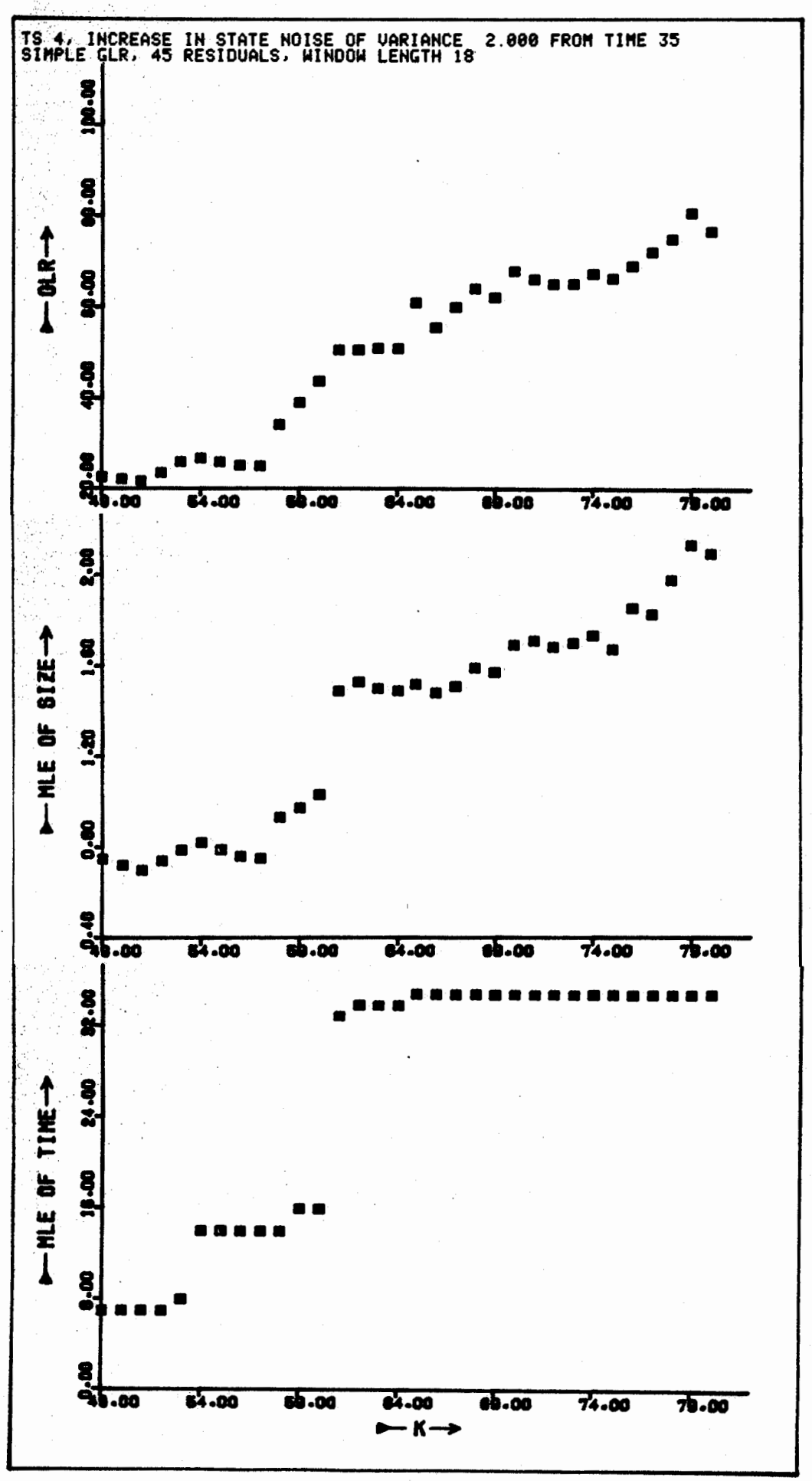

TS 4, INCREASE IN STATE NOISE OF VARIANCE 2.000 FROM TIME 35
SIMPLE GLR, 45 RESIDUALS, WINDOW LENGTH 18
GLR
MLE OF SIZE
MLE OF TIME
K
49.00
54.00
59.00
64.00
69.00
74.00
79.00

fig. 29

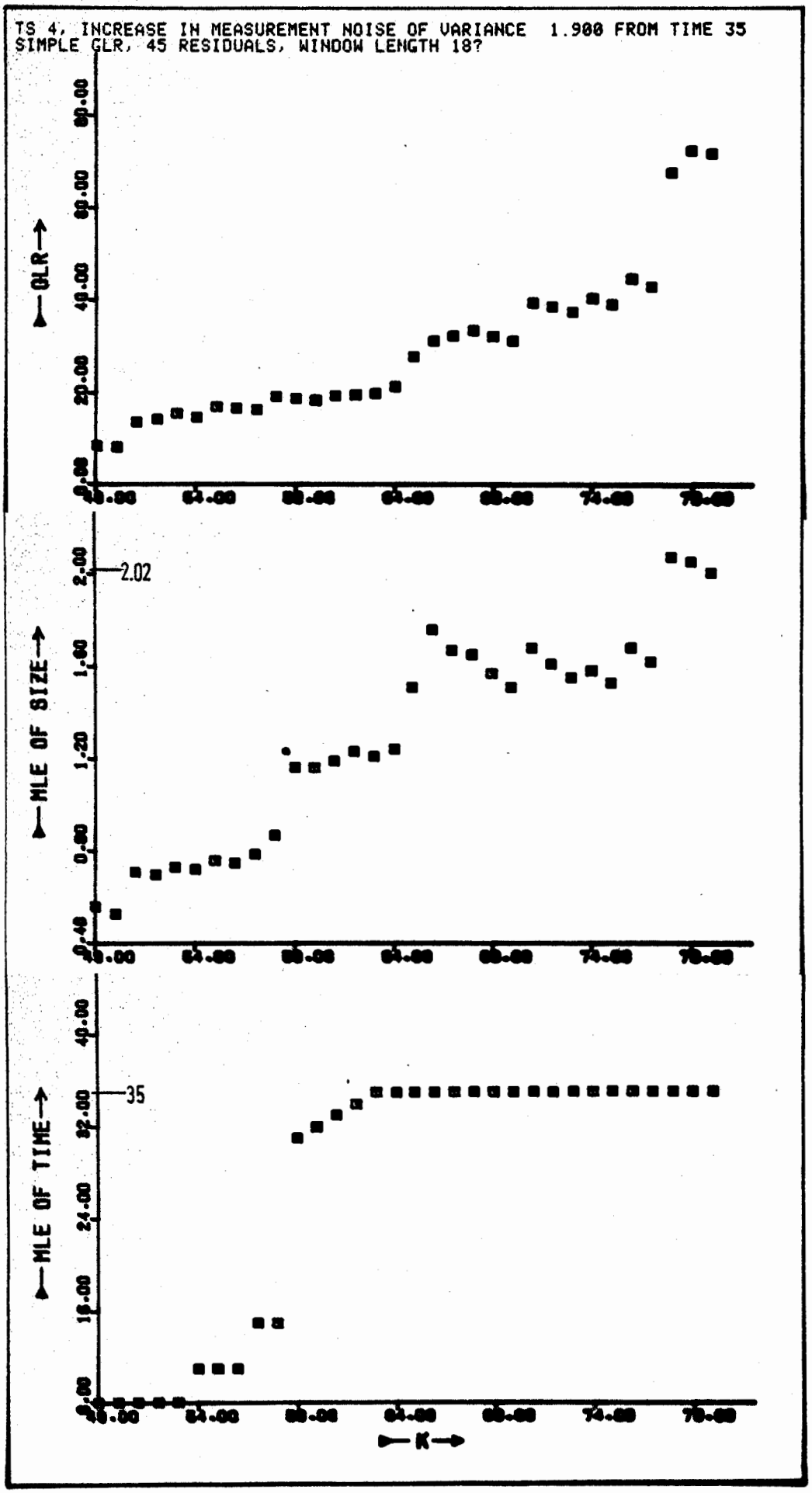

TS 4, INCREASE IN MEASUREMENT NOISE OF VARIANCE 1.900 FROM TIME 35
SIMPLE GLR, 45 RESIDUALS, WINDOW LENGTH 18?
GLR
MLE OF SIZE
2.02
MLE OF TIME
35
K

TABLE 16

TEST SYSTEM 4, INCREASED MEASUREMENT NOISE OF SIZE 1.9 FROM TIME 35

MULTIPLE GLR, 45 RESIDUALS, WINDOW LENGTH 18

GLR	SIZE	TIME
5.48,	.63,	7
4.79,	.61,	7
10.15,	.75,	7
9.44,	.74,	7
10.35,	.78,	8
9.55,	.79,	4
10.96,	.83,	14
11.20,	.82,	14
10.81,	.80,	14
14.06,	.87,	14
14.24,	.85,	14
13.35,	.84,	15
13.09,	1.15,	24
14.08,	1.17,	24
13.32,	1.13,	24
14.92,	1.15,	24
23.08,	1.32,	24
30.15,	1.41,	24
29.82,	1.43,	24
30.73,	1.42,	34
29.73,	1.36,	34
28.91,	1.31,	34
36.01,	1.49,	34
34.69,	1.45,	34
33.66,	1.40,	34
36.04,	1.41,	34
35.20,	1.39,	34
41.68,	1.49,	34
40.63,	1.44,	34
62.17,	2.00,	37
63.77,	2.13,	37
62.80,	2.08,	37

SIMPLE GLR, 45 RESIDUALS, WINDOW LENGTH 18

GLR	SIZE	TIME
8.73,	.56,	8
8.43,	.53,	8
13.87,	.71,	8
14.48,	.70,	8
15.81,	.73,	8
14.93,	.72,	11
17.19,	.76,	11
16.87,	.75,	11
16.60,	.79,	15
19.46,	.87,	15
18.93,	1.16,	31
18.70,	1.16,	32
19.63,	1.19,	33
19.80,	1.23,	34
20.10,	1.21,	35
21.53,	1.24,	35
27.87,	1.51,	35
31.22,	1.76,	35
32.46,	1.67,	35
33.46,	1.65,	35
32.26,	1.57,	35
31.21,	1.51,	35
39.46,	1.68,	35
38.64,	1.61,	35
37.46,	1.55,	35
40.36,	1.58,	35
39.14,	1.53,	35
44.71,	1.68,	35
42.99,	1.62,	35
67.64,	2.07,	35
72.35,	2.05,	35
71.71,	2.00,	35

CHAPTER VI: *GENERAL CONCLUSIONS*

In this thesis, fault monitoring schemes for discrete, linear stochastic systems subject to faults in their parameters are discussed. The proposed schemes consist of a state estimation Kalman filter based on a no-fault hypothesis and a fault monitoring system based on simple statistical tests and GLR hypothesis testing detectors. Once a fault is detected, the filter can be adjusted using the information of the fault monitoring scheme. The analysis and simulation tests were carried out on first order systems, though multivariable and time-varying systems can be similarly treated, at least as far as the analysis is concerned.

An overall system, incorporating fault monitoring schemes, can be extremely complex and a final choice depends heavily on the particular application. Such issues as available computational facilities and cost of implementation enter in a crucial way into the design decision. For this reason the proposed fault monitoring schemes range from the simple to the complex, thus being suited for most real applications.

The test results obtained for the proposed detection partial-isolation algorithms show that it is possible to detect and partially isolate the nature of the occurred fault with a set of simple detection algorithms based on statistical inference. Furthermore, if the class of the faults is suitably reduced, complete fault isolation is possible.

The performance of the statistical tests employed, confirm their theoretical behaviour. The simple nature of the statistics involved and their known probability distributions under the null hypothesis, permit a detailed analysis of the performance of each test for chosen

sample size and probabilities of false alarm and correct detection. Conversely, given a set of adequate performance criteria, such as specified probability of false alarm, the remaining parameters can be optimally selected. The tests were characterised by significant detection delay times, but as discussed in Section V.2.1.1, this was due to the large sample size used. This implies that in practice a tradeoff study between detection delay time and probability of false alarms has to be carried out.

The implementation and characteristics for the GLR tests are different in the cases of faults in the mean and variance parameters of the noise sequences.

In the case of bias in the plant or measurements, the implementation of the GLR algorithm is quite straightforward, due mainly to the fact that the maximisation of the likelihood function under the alternative hypothesis of fault may be obtained analytically. Further the MLE of the fault size is explicitly related to the MLE of the time of fault occurrence. The probability distributions of the GLR statistics under both the null and alternative hypotheses are well known distributions, a fact which again permits a detailed analysis of the performance characteristics of the GLR test. Thus, the functional relationship between the probability of false alarm, the probability of correct detection, the threshold value and the sample size can be used for the determination of optimal strategies according to prespecified performance criteria. The simulation results obtained in the case of additional plant or measurement noise confirm the theoretical predictions. These results indicate the potential usefulness of the method, as high correct detection rates and small false alarm rates were obtained, even though the size of the simulated faults was well below the 10σ levels used by

other researchers in the field *[36]*. The GLR test performed well in the case where it had to distinguish between two types of faults, making consistently correct decisions.

The analysis and test results in the cases of additional noise or furthermore in the case of a change in the parameters ϕ or η, show that the problems of detection, isolation and identification present a considerable challenge from the standpoints of theory and practical implementation . The problems arise mainly from the form of the likelihood function which has to be maximised by numerical techniques. The strong nonlinear dependence of the likelihood equation on the fault size requires an initial guess of the region containing the maximum or discretization of the interval in which the fault size is constrained. Thus the problems which arise for scalar systems may be overcome, but the extension to multivariate systems will not be so straightforward. Nevertheless, in applications where monitoring for additional noise is essential the proposed scheme may serve in a number of ways. Firstly, the values of the likelihood function at different points in the parameter space may be used to make relative statements about the likelihood of one parameter value versus another. Secondly, the performance of the filter in the event of faults can be analysed using the equations developed, and the overall system designed so that the effects from possible faults minimised. Thirdly, attempts to approximate the likelihood function surface by some quadratic surface may be made in specific cases and where conditions of the problem permit. Finally, storage and computing requirements may be reduced by using GLR tests in conjunction with simpler statistical tests, even though overall performance may suffer. In this manner, a bank of tests may be implemented that range from the very simple to the very complex.

The simulation results in the cases of additional state and measurement noise were satisfactory, with correct identification of the the time of fault occurrence and acceptable estimates of the fault size. In this case the range of the fault size was constrained and the maximisation of the likelihood function was performed under this constraint. Good estimates of the fault size, however, were obtained using large sample sizes resulting in increased computing time. Threshold values and sample sizes must be set experimentally using simulation techniques, since the probability distribution of the GLR statistic under both hypotheses on test are not known.

Since no similar techniques are known to the author to exist, no direct comparison is possible. In general, however, residual-based fault monitoring schemes offer several advantages over other methods: they can be adopted to utilize the residuals of an existing filter in a number of different ways, offering various tradeoffs between performance and complexity; no performance degradation is suffered prior to the occurrence of a fault, unlike other methods which use suboptimal filters.

Further topics for research

The develpment of fault monitoring schemes is still new and the recommendations for further study numerous. Since the field is also closely related to statistics, the suggestions contain interesting problems for statistical research.

Firstly, the practical implementation of the proposed monitoring schemes to multivariable, time-varying systems can be investigated along the lines of the present work. As mentioned earlier, the extension in the case of state or measurement bias is simple, but difficulties exist in implementing schemes for additional noise or

changes in the transition matrices Φ and H. The problem of monitoring for multiple faults or distinguishing between more than two possible types of faults provide ground for statistical analysis, as indeed the distributions of the GLR statistics under the hypotheses on test.

Different classes of faults, other than additive, or alternative modelling should also be researched.

The class of linear systems with gaussian, uncorrelated disturbances cover only a small section of real systems and research on fault monitoring for various variations of the process model must be carried out. The extension to non-linear stochastic systems should also be considered.

These and other issues, notably the application of the proposed methods to real systems, await future investigation.

APPENDICES

APPENDIX I.1

Theorem: The state, measurement, filter estimate and residual sequences for systems represented by (I.1)-(I.2) which are subject to possible additive type faults in the values of their parameters, may be written as:

$$x(k) = h_x(k,\theta,\Delta p) + x_o(k) \qquad \text{(III.11)}$$

$$y(k) = h_y(k,\theta,\Delta p) + y_o(k) \qquad \text{(III.12)}$$

$$\hat{x}(k/k) = f(k,\theta,\Delta p) + \hat{x}_o(k/k) \qquad \text{(III.13)}$$

$$\gamma(k) = g(k,\theta,\Delta p) + \gamma_o(k) \qquad \text{(III.15)}$$

Proof: The proof will be by induction. Suppose (III.11)-(III.15) hold for time k. At k+1, $\hat{x}(k+1/k+1)$, $\gamma(k+1)$ are calculated by the Kalman filter as,

$$\begin{aligned}\gamma(k+1) &= y(k+1) - \eta\phi\hat{x}(k/k) \\ &= y_o(k+1) + h_y(k+1,\theta,\Delta p) - \eta\phi\{\hat{x}_o(k/k)+f(k,\theta,\Delta p)\} \\ &= \gamma_o(k+1) + h_y(k+1,\theta,\Delta p) - \eta\phi f(k,\theta,\Delta p) \qquad \text{(A.1)}\end{aligned}$$

$$\begin{aligned}\text{and, } \hat{\hat{x}}(k+1/k+1) &= \phi\hat{x}(k/k) + K(k+1)\gamma(k+1) \\ &= \phi\{\hat{x}_o(k/k)+f(k,\theta,\Delta p)\} + K(k+1)\{\gamma_o(k+1)+h_y(k+1,\theta,\Delta p) - \\ &\quad \eta\phi f(k,\theta,\Delta p)\} \\ &= \hat{x}_o(k+1/k+1) + \phi f(k,\theta,\Delta p) + \\ &\quad + K(k+1)\{h_y(k+1,\theta,\Delta p) - \eta\phi f(k,\theta,\Delta p)\} \qquad \text{(A.2)}\end{aligned}$$

where the subscript O denotes the value of the parameter that is obtained if no fault ocuurs. Equations (A.1), (A.2) may be rewritten,

$$\gamma(k+1) = \gamma_o(k+1) + g(k+1,\theta,\Delta p)$$

$$\hat{x}(k+1,k+1) = \hat{x}_o(k+1/k+1) + f(k+1,\theta,\Delta p)$$

where,

$$g(k+1,\theta,\Delta p) = h_y(k+1,\theta,\Delta p) - \eta\phi f(k,\theta,\Delta p)$$

$$f(k+1,\theta,\Delta p) = \phi f(k,\theta,\Delta p) + K(k+1)g(k+1,\theta,\Delta p)$$

At $k=\theta$, since the fault has not affected $\hat{x}(\theta-1/\theta-1)$

$$\gamma(\theta) = y(\theta) - \eta\phi\hat{x}(\theta-1/\theta-1)$$

$$= y_o(\theta) + h_y(\theta,\theta,\Delta p) - \eta\phi\hat{x}(\theta-1/\theta-1)$$

$$= \gamma_o(\theta) + h_y(\theta,\theta,\Delta p)$$

and $$\hat{x}(\theta/\theta) = \phi\hat{x}(\theta-1/\theta-1) + K(\theta)\gamma(\theta)$$

$$= \hat{x}_o(\theta/\theta) + K(\theta)h_y(\theta,\theta,\Delta p)$$

Hence,

$$\gamma(\theta) = \gamma_o(\theta) + g(\theta,\theta,\Delta p)$$

$$\hat{x}(\theta/\theta) = \hat{x}_o(\theta/\theta) + f(\theta,\theta,\Delta p)$$

where,

$$g(\theta,\theta,\Delta p) = h_y(\theta,\theta,\Delta p)$$

$$f(\theta,\theta,\Delta p) = K(\theta)g(\theta,\theta,\Delta p)$$

This completes the proof.

I.1(a): *Mean value of residuals in steady state following a step bias in the state.*

The mean value of the residuals in the event of a step bias in the state is given by (III.63) as:

$$\bar{\gamma}(k) = \sum_{i=\theta}^{k} g_a(k,i)\nu_x$$

where the g_a are given by (III.22)-(III.24) as:

$$g_a(i,j) = \eta\{\phi^{i-j} - \phi f_a(i-1,j)\} \qquad \text{(III.22)}$$

$$f_a(i,j) = K(i)g_a(i,j) + \phi f_a(i-1,j);\ i \geq j \qquad \text{(III.23)}$$

$$g_a(i,j) = f_a(i,j) = 0;\ i<j \qquad \text{(III.24)}$$

Substituting for $g_a(i,j)$ in (III.23):

$$\begin{aligned} f_a(i,j) &= K(i)\eta\{\phi^{i-j} - \phi f_a(i-1,j)\} + \phi f_a(i-1,j) \\ &= (1-K(i)\eta)\phi f_a(i-1,j) + K(i)\eta\phi^{i-j} \end{aligned}$$

Assuming that the filter has reached steady state, i.e. $K(i)=K$, $i \geq \theta$ (III.22)-(III.24) depend on $(i-j)$ only. Hence,

$$f_a(i-j) = sf_a(i-1-j) + s_{i-j} \qquad \text{(A.3)}$$

where,
$$s = (1-K\eta)\phi \qquad \text{(A.4)}$$

$$s_{i-j} = K\eta\phi^{i-j} \qquad \text{(A.5)}$$

In view of assumptions of uniform complete controllability and uniform complete observability and Theorem 1,

$$\lim_{k\to\infty} s^k = 0 \qquad \text{(A.6)}$$

and since the system is stable, $|\phi|<1$, and,

$$\lim_{(i-j)\to\infty} s_{i-j} = 0 \qquad (A.7)$$

Equation (A.3) is a difference equation in f_a, similar to (I.1), with initial conditions, obtained from (III.23) for i=j,

$$f_a(0) = K\eta$$

Its solution is *[47]*:

$$\begin{aligned} f_a(m) &= s^m f_a(0) + \sum_{i=0}^{m-1} s^{m-i-1} s_{i+1} \\ &= s^m f_a(0) + s^{m-1} \sum_{i=0}^{m-1} s^{-i} s_{i+1} \\ &= s^m K\eta + s^{m-1} K\eta\phi \sum_{i=0}^{m-1} s^{-i}\phi^i \end{aligned}$$

Hence,

$$\begin{aligned} f_a(m) &= K\eta\{s^m + \phi \frac{s^m-\phi^m}{s-\phi}\} \\ &= s^m(K\eta-1) + \phi^m \qquad (A.8) \end{aligned}$$

and,

$$\begin{aligned} g_a(m) &= \eta\{\phi^m - \phi\{s^{m-1}(K\eta-1) + \phi^{m-1}\}\} \\ &= \eta s^m \qquad (A.9) \end{aligned}$$

$$\therefore \quad \sum_{m=0}^{\infty} g_a(m) = \frac{\eta}{1-s} \qquad (A.10)$$

and,

$$\bar{\gamma}(k) = \frac{\eta}{1-s}\nu_x \; ; \quad k>>\theta \qquad (A.11)$$

I.1(b): *Steady state value of g_b in the event of a change in ϕ.*

Equations (III.34)-(III.36) describe the recursive relations for g_b in the case of a step change in the state transition coefficient ϕ. It may be seen that these are the same as (III.22)-(III.24) except only for the form of the h_x functions, which is ϕ^{i-j} for the case of a step bias and $(\phi+\Delta\phi)^{i-j}$ for the case of a step change in ϕ. Therefore, the analysis of the previous Section can be carried over in this case, having made the appropriate changes. Thus, equation (A.3) becomes:

$$f_b(i-j,\Delta\phi) = sf_b(i-j,\Delta\phi) + \bar{s}_{i-j} \tag{A.12}$$

where, $$\bar{s}_{i-j} = K\eta(\phi+\Delta\phi)^{i-j}$$

$$= K\eta\bar{\phi}^{i-j} \tag{A.13}$$

Since it is assumed that $|\bar{\phi}|<1$,

$$\lim_{(i-j)\to\infty} \bar{s}_{i-j} = 0$$

and equation (A.8) becomes:

$$f_b(m,\Delta\phi) = K\eta\{s^m + \bar{\phi}\,\frac{s^m-\bar{\phi}^m}{s-\bar{\phi}}\} \tag{A.14}$$

$$\therefore \quad g_b(m,\Delta\phi) = \eta\{\bar{\phi}^m - \eta K\phi\{s^{m-1} + \bar{\phi}\,\frac{s^{m-1}-\bar{\phi}^{m-1}}{s-\bar{\phi}}\}\}$$

$$= (\eta+\frac{\eta K\phi}{s-\bar{\phi}})\bar{\phi}^m - \eta K\phi(1+\frac{\bar{\phi}}{s-\bar{\phi}})s^{m-1} \tag{A.15}$$

Thus, since both $|\bar{\phi}|<1$ and $|s|<1$,

$$\lim_{m\to\infty} g_b(m,\Delta\phi) = 0$$

I.1(c): *Correlation of residuals in steady state under additional plant noise.*

The correlation of residuals in the event of additional plant noise is given by (III.77) as,

$$\mathrm{cov}[\gamma(k),\gamma(m)] = c(k,m) + \sum_{i=\theta}^{\lambda} g_c(k,i)\,g_c(m,i)\,s_x$$

where $c(k,m)$ is defined by (II.2), $\lambda=\min\{k,m\}$ and the g_c are defined by (III.39)-(III.41) which are the same as (III.22)-(III.24). Hence, using (A.9) and letting $\lambda=k$:

$$\sum_{i=\theta}^{k} g_c(k,i)\,g_c(m,i) = \sum_{i=\theta}^{k} g_c(k-i)\,g_c(m-i)$$

$$= \sum_{i=\theta}^{k} \eta s^{k-i}\,\eta s^{m-i}$$

$$= \eta^2 s^{k+m} \sum_{i=\theta}^{k} s^{-2i}$$

$$= \eta^2 s^{k+m-2\theta}\left\{\frac{1-s^{-2(k-\theta+1)}}{1-s^{-2}}\right\} \qquad \text{(A.16)}$$

Since equation (A.16) does not depend on (k-m) only, it follows that the residual sequence is not stationary in the event of additional plant noise. However, for $k+m>>2\theta$,

$$\sum_{i=\theta}^{k} g_c(k-i)\,g_c(m-i) = -\frac{\eta^2 s^{k-m-2}}{1-s^{-2}}$$

$$= s^{m-k}\left\{\frac{\eta^2}{1-s^2}\right\} \qquad \text{(A.17)}$$

Hence,

$$\mathrm{cov}[\gamma(k),\gamma(m)] = c(k,m) + s^{m-k}\,\frac{\eta^2}{1-s^2}\,s_x;\ k+m>>2\theta \qquad \text{(A.18)}$$

I.1(d): *Mean value of residuals in steady state following a step bias in the observations.*

The mean value of the residual sequence in the event of a step bias in the observations is given by (III.80) as:

$$\bar{\gamma}(k) = \sum_{i=\theta}^{k} g_d(k,i)\nu_y$$

where the g_d are defined by (III.42)-(III.45) as:

$$g_d(i,j) = -\eta\phi f_d(i-1,j); \quad i>j \tag{III.42}$$

$$f_d(i,j) = K(i)g_d(i,j) + \phi f_d(i-1,j); \quad i\geqslant j \tag{III.43}$$

$$g_d(i,i) = 1 \tag{III.44}$$

$$g_d(i,j) = f_d(i,j) = 0; \quad i<j \tag{III.45}$$

If K(i) has reached a constant value, K, (III.42)-(III.45) can be written in stationary form as:

$$g_d(m) = -\eta\phi f_d(m-1); \quad m>0 \tag{A.19}$$

$$f_d(m) = Kg_d(m) + \phi f_d(m-1); \quad m\geqslant 0 \tag{A.20}$$

$$g_d(0) = 1 \tag{A.21}$$

$$g_d(m) = f_d(m) = 0; \quad m<0 \tag{A.22}$$

Substituting (A.19) in (A.20),

$$\begin{aligned} f_d(m) &= -K\eta\phi f_d(m-1) + \phi f_d(m-1) \\ &= (1-K\eta)\phi f_d(m-1) \\ &= sf_d(m-1) \\ &= s^m f_d(0) = s^m K \end{aligned} \tag{A.23}$$

and $g_d(m) = -\eta\phi s^{m-1}K$, hence,

$$\sum_{i=\theta}^{k} g_d(k,i) = \sum_{i=\theta}^{k} g_d(k-i) = \sum_{i=\theta}^{k-1} \{-\eta\phi s^{k-i-1}\}K + 1$$

$$= 1 - \eta\phi K s^{k-\theta-1} \frac{1-s^{\theta-k}}{1-s^{-1}}$$

$$= 1 + \eta\phi K s^{-1} \frac{1-s^{k-\theta}}{1-s^{-1}}$$

and for $k>>\theta$, $= 1 + \frac{\eta\phi K}{s-1}$ (A.24)

Hence,

$$\bar{\gamma}(k) = \left\{1 + \frac{\eta\phi K}{s-1}\right\} \nu_y \; ; \quad k>>\theta \tag{A.25}$$

I.1(e): *Correlation of residuals in steady state under additional measurement noise.*

The correlation of the Kalman filter residuals in the event of additional measurement noise is given by (III.87) as:

$$\text{cov}[\gamma(k),\gamma(m)] = c(k,m) + \sum_{i=\theta}^{\lambda} g_f(k,i)\, g_f(m,i)\, s_y$$

where $\lambda=\min\{k,m\}$, and g_f is defined similarly to g_d by (III.42)-(III.45). Hence, letting $\lambda=k\neq m$:

$$\sum_{i=\theta}^{k} g_f(k,i)\, g_f(m,i) = \sum_{i=\theta}^{k} g_f(k-i)\, g_f(m-i)$$

$$= \sum_{i=\theta}^{k-1} (-\eta\phi K s^{k-i-1})(-\eta\phi K s^{m-i-1}) + g_f(0)\, g_f(m-k)$$

$$= (\eta\phi K)^2 s^{k+m-2} \sum_{i=\theta}^{k-1} s^{-2i} + g(m-k)$$

$$= (\eta\phi K)^2 \, s^{k+m-2} s^{-2\theta} \frac{1-s^{-2(k-\theta)}}{1-s^{-2}} - \eta\phi K s^{m-k-1}$$

and if $k+m>>2\theta$, $= -\eta\phi K s^{m-k-1} \left\{\frac{\eta\phi K s^{-1}}{1-s^{-2}} + 1\right\}$

$$= -s^{m-k} \left\{\frac{(\eta\phi K)^2}{s^2-1} + \eta\phi K s^{-1}\right\}$$

Hence,

$$\mathrm{cov}[\gamma(k),\gamma(m)] = -s^{m-k}\left\{\frac{(\eta\phi K)^2}{s^2-1} + \eta\phi K s^{-1}\right\} s_y; \quad k\neq m,\ k+m>>2\theta \qquad \text{(A.26)}$$

When k=m,

$$\sum_{i=\theta}^{k} g_f^2(k,i) = \sum_{i=\theta}^{k} (-\eta\phi K s^{k-i-1})^2 + g^2(0)$$

$$= (\eta\phi K)^2 s^{2(k-1)} s^{-2\theta} \frac{1-s^{-(k-\theta)}}{1-s^{-2}} + 1$$

If $k>>\theta$, $= 1 - \frac{(\eta\phi K)^2}{s^2-1}$

Hence,

$$\mathrm{var}[\gamma(k)] = \left\{1 - \frac{(\eta\phi K)^2}{s^2-1}\right\} s_y + c(k,k) \qquad \text{(A.27)}$$

I.2: *Results for the multivariable case.*

The multivariable equivalent of the system described by (I.1)-(I.2) is given by (III.95)-(III.96). The state estimate and residual vector sequences for this system are calculated via the multidimensional Kalman filter by:

$$\underline{\gamma}(k) = \underline{y}(k) - H(k)\hat{\underline{x}}(k/k-1)$$

$$= \underline{y}(k) - H(k)\Phi(k,k-1)\hat{\underline{x}}(k-1/k-1) \qquad \text{(A.28)}$$

$$\underline{x}(k/k) = \hat{\underline{x}}(k/k-1) + K(k)\underline{\gamma}(k) \qquad \text{(A.29)}$$

Following the lines of proof for the theorem of Appendix I.1, it may be shown that in the multivariable case the following equivalent theorem holds:

Theorem: The state, measurement, filter estimate and residual vector sequences for the multivariable system described by (III.95)-(III.96), which are subject to additive type faults in the values of their parameters may be written as:

$$\underline{x}(k) = \underline{h}_x(k,\theta,\Delta\underline{p}) + \underline{x}_o(k) \qquad \text{(A.30)}$$

$$\underline{y}(k) = \underline{h}_y(k,\theta,\Delta\underline{p}) + \underline{y}_o(k) \qquad \text{(A.31)}$$

$$\hat{\underline{x}}(k/k) = \underline{f}(k,\theta,\Delta\underline{p}) + \hat{\underline{x}}_o(k/k) \qquad \text{(A.32)}$$

$$\underline{\gamma}(k) = \underline{g}(k,\theta,\Delta\underline{p}) + \underline{\gamma}_o(k) \qquad \text{(A.33)}$$

where the vector functions $\underline{h}_x$, $\underline{h}_y$ represent the effect of the fault on the state and measurement sequences respectively and $\underline{f}$, $\underline{g}$ are computed from the following recursive vector equations:

$$\underline{g}(k,\theta,\Delta\underline{p}) = \underline{h}_y(k,\theta,\Delta\underline{p}) - H(k)\Phi(k,k-1)\underline{f}(k-1,\theta,\Delta\underline{p}) \quad (A.34)$$

$$\underline{f}(k,\theta,\Delta\underline{p}) = K(k)\underline{g}(k,\theta,\Delta\underline{p}) + \Phi(k,k-1)\underline{f}(k-1,\theta,\Delta\underline{p}) \; ; \; k\geq\theta \quad (A.35)$$

$$\underline{g}(k,\theta,\Delta\underline{p}) = \underline{f}(k,\theta,\Delta\underline{p}) = \underline{0}; \quad k<\theta \quad (A.36)$$

Hence, in the case of additional measurement noise, the effect on the residual sequence can be similarly found, as in the case of scalar systems, by considering the effect of successive, corresponding type I faults.

Therefore, (III.56) becomes:

$$\underline{\gamma}(k) = \underline{\gamma}_o(k) + \sum_{i=\theta}^{k} G(k,i)\zeta_y(i) \quad (A.37)$$

where the matrices G(k,i) are recursively computed from:

$$G(i,j) = -H(i)\Phi(i,i-1)F(i-1,j); \quad i>j \quad (A.38)$$

$$F(i,j) = K(i)G(i,j) + \Phi(i,i-1)F(i-1,j); \quad i\geq j \quad (A.39)$$

$$G(i,i) = I \quad (A.40)$$

$$F(i,j) = G(i,j) = 0; \quad i<j \quad (A.41)$$

Substituting (A.38) in (A.39),

$$\begin{aligned} F(i,j) &= -K(i)H(i)\Phi(i,i-1)F(i-1,j) + \Phi(i,i-1)F(i-1,j) \\ &= (I-K(i)H(i))\Phi(i,i-1)F(i-1,j) \\ &= \prod_{m=j+1}^{i} (I-K(m)H(m))\Phi(m,m-1)\; F(j,j) \\ &= \prod_{m=j+1}^{i} (I-K(m)H(m)\Phi(m,m-1)\; K(j) \end{aligned} \quad (A.42)$$

where the matrix product is defined by:

$$\begin{aligned} \prod_{m=k}^{n} S_i &= S_n\, S_{n-1} \cdots S_k \; ; \; n\geq k \\ &= 0; \; n<k \end{aligned}$$

Hence,

$$G(i,j) = -H(i)\Phi(i,i-1)\prod_{m=j+1}^{i-1}(I-K(m)H(m))\Phi(m-m-1)K(j) \quad (A.43)$$

To calculate the correlation between residuals $\underline{\gamma}(k)$ and $\underline{\gamma}(m)$, the sum

$$\Sigma(k,m,\theta) \triangleq \sum_{i=\theta}^{\lambda} G(k,i)S(i)G^T(m,i)$$

must be calculated. Let,

$$\Gamma(i,j) = \prod_{m=j+1}^{i-1}(I-K(m)H(m))\Phi(m,m-1)K(j)$$

then, $G(i,j) = -H(i)\Phi(i,i-1)\Gamma(i,j)$.

Let $k=\lambda$,

$$\Sigma(k,m,\theta) = \sum_{i=\theta}^{k} H(k)\Phi(k,k-1)\Gamma(k,i)S(i)\Gamma^T(m,i)\Phi^T(m,m-1)H^T(m)$$

$$= H(k)\Phi(k,k-1)\left\{\sum_{i=\theta}^{k}\Phi(k,i)S(i)\Phi^T m,i)\right\}\Phi^T(m,m-1)H^T(m)$$

If the system is uniformly completely observable and uniformly completely controllable,

$$\left|\left|\sum_{m=O}^{k}(I-K(m)H(m))\Phi(m,m-1)\right|\right| \to O, \text{ exponentially as } k\to\infty.$$

Therefore G(i,j) described by (A.43), will tend to zero, and consequently $\Sigma(k,m,\theta)$ will tend to a constant matrix.

If a time invariant system is considered and the filter is in steady state,

$$G(i,j) = -H\Phi\{(I-KH)\Phi\}^{i-j}K$$

and $$\Sigma(k,m,\theta) = H\Phi\left\{\sum_{i=\theta}^{k}\{(I-KH)\Phi\}^{k-i}KSK^T\{\{(I-KH)\Phi\}^{m-i}\}^T\right\}\Phi^T H^T$$

Consider the sum,

$$\sum_{i=\theta}^{k} W^{k-i} T (W^T)^{m-1} = \{\sum_{i=\theta}^{k} W^{k-1} T (W^T)^{k-i}\} (W^T)^{m-k} = S_{k,\theta}$$

where, $W = (I-KH)\Phi$

$T = KSK^T$

Expanding the sum yields:

$$T + WTW^T + \ldots + W^{k-\theta} T (W^T)^{k-\theta}$$

$$= T + W\{T + \ldots + W^{k-\theta-1} T (W^T)^{k-\theta-1}\}W^T$$

If $k>>\theta$,

$$S_{\infty,\theta} = T + WS_{\infty,\theta}W^T \qquad \text{(A.44)}$$

This is a linear equation in the elements of $S_{\infty,\theta}$ which can be solved from a set of simultaneous linear equations. Therefore, in steady state under additional measurement noise:

$$\Sigma(k,m,\theta) = H\Phi S_{\infty,\theta}(W^T)^{m-k} \qquad \text{(A.45)}$$

I.3: *Iterative calculation of serial correlation.*

The serial correlation of lag m is defined by:

$$\hat{c}_m^j = \frac{1}{n} \sum_{k=m_1}^{j} \{\gamma(k) - \hat{\gamma}^{i,j}\}\{\gamma(k-m) - \hat{\gamma}^{i,j}\}$$

where the superscript j denotes that the statistic is calculated from the sample $\underline{\gamma}^{i,j}$ and $m_1 = j-n+m+1$. The iteration on m will be considered first.

$$\hat{c}_{m-1}^j = \frac{1}{n} \sum_{k=m_1-1}^{j} \{\gamma(k) - \hat{\gamma}^{i,j}\}\{\gamma(k-m+1) - \hat{\gamma}^{i,j}\}$$

$$= \frac{1}{n} \sum_{k=m_1-1}^{j} \{\gamma(k)\gamma(k-m+1) - \hat{\gamma}^{i,j}(\gamma(k-m+1)+\gamma(k)) - (\hat{\gamma}^{i,j})^2\}$$

Similarly,

$$\hat{c}_m^j = \frac{1}{n} \sum_{k=m_1}^{j} \{\gamma(k)\gamma(k-m) - \hat{\gamma}^{i,j}(\gamma(k-m)+\gamma(k)) - (\hat{\gamma}^{i,j})^2\}$$

Comparing terms,

$$\hat{\gamma}^{i,j} \sum_{k=m_1+1}^{j} \gamma(k) = \hat{\gamma}^{i,j} \sum_{k=m_1}^{j} \gamma(k) - \hat{\gamma}^{i,j}\gamma(m_1)$$

$$= q_m^j \; \hat{\gamma}^{i,j}$$

$$\hat{\gamma}^{i,j} \sum_{k=m_1+1}^{j} \gamma(k-m) = \hat{\gamma}^{i,j} \sum_{k=m_1}^{j} \gamma(k-m) - \hat{\gamma}^{i,j}\gamma(k-m+1)$$

$$= p_m^j \; \hat{\gamma}^{i,j}$$

$$\sum_{k=m_1+1}^{j} (\hat{\gamma}^{i,j})^2 = \sum_{k=m_1}^{j} (\hat{\gamma}^{i,j})^2 - (\hat{\gamma}^{i,j})^2 = (n-m)(\hat{\gamma}^{i,j})^2$$

Therefore, $\hat{c}^j_m$ can be calculated from $\hat{c}^j_{m-1}$ using:

$$\hat{c}^j_m = \frac{1}{n}\left\{(n-m)\left(\hat{\gamma}^{i,j}\right)^2 - (q^j_m+p^j_m)\hat{\gamma}^{i,j} + \sum_{k=m_1+1}^{j}\gamma(k)\gamma(k-m)\right\}$$

where,

$$q^j_m = q^j_{m-1} - \gamma(j-n+m)$$

$$p^j_m = p^j_{m-1} - \gamma(j-n+1)$$

initialised by:

$$q^j_0 = p^j_0 = \sum_{k=m_1}^{j}\gamma(k) = n\hat{\gamma}^{i,j}$$

It remains to establish a recursion of $\hat{c}^j_0$ from $\hat{c}^{j-1}_0$.

The sample mean can be calculated iteratively, using

$$\hat{\gamma}^{i,j} = \hat{\gamma}^{i-1,j-1} + \frac{1}{n}\{\gamma(j)-\gamma(i-1)\}$$

Then,

$$\hat{c}^j_0 = \sum_{k=i}^{j}\{\gamma(k)-\hat{\gamma}^{i,j}\}^2$$

$$\hat{c}^{j-1}_0 = \sum_{k=i-1}^{j}\{\gamma(k)-\hat{\gamma}^{i-1,j-1}\}^2$$

Comparing terms,

$$\hat{c}^j_0 = \hat{c}^{j-1}_0 + \frac{1}{n}\{\gamma^2(j)-\gamma^2(i-1)\} - \{\gamma(j)-\gamma(i-1)\}^2 - 2\hat{\gamma}^{i-1,j-1}\{\gamma(j)-\gamma(i-1)\}$$

Let, $a^j = \frac{1}{n}\{\gamma(j)-\gamma(i-1)\}$ then

$$\hat{c}^j_0 = \hat{c}^{j-1}_0 + \frac{1}{n}\{\gamma^2(j)-\gamma^2(i-1)\} - a^j(a^j+2\hat{\gamma}^{i-1,j-1})$$

I.4: *Maximisation of the likelihood equation*

The likelihood equation is given by (IV.20) as:

$$f(\theta,s) = -2\ln\pi(j,\theta-1) + \ln|C| + \underline{\gamma}^T C^{-1}\underline{\gamma}$$

where $C = E + sC^*$

The function $f(\theta,s)$ is highly nonlinear in s, the degree of non-linearity depending on the dimension of C. It will in general have multiple minima. Its maximisation, even in simple cases, is not possible analytically. Consider, for example a problem in two dimensions for fixed θ.

$$C = \begin{bmatrix} e_1 & \\ & e_2 \end{bmatrix} + s \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{11} \end{bmatrix} = \begin{bmatrix} sc_{11}+e_1 & sc_{12} \\ sc_{12} & sc_{22}+e_2 \end{bmatrix}$$

$$\therefore |C| = (sc_{11}+e_1)(sc_{22}+e_2) - s^2c_{12}^2$$

$$= a_2s^2 + a_1s + a_0$$

where,

$$a_2 = c_{11}c_{22} - c_{12}^2$$
$$a_1 = c_{11}e_2 + c_{22}e_1$$
$$a_0 = e_1e_2$$

and,

$$C^{-1} = \begin{bmatrix} sc_{11}+e_1 & sc_{12} \\ sc_{12} & sc_{22}+e_2 \end{bmatrix}^{-1}$$

$$= \frac{1}{|C|}\begin{bmatrix} sc_{22}+e_2 & -sc_{12} \\ -sc_{12} & sc_{11}+e_1 \end{bmatrix}$$

Hence,

$$\underline{\gamma}^T C^{-1} \underline{\gamma} = \frac{1}{|C|} \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} sc_{22}+e_2 & -sc_{12} \\ -sc_{12} & sc_{11}+e_1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

$$= \frac{1}{|C|}(b_1 s + b_0)$$

where b_1, b_0 are appropriate constants. Stationary points will occur if,

$$\frac{\partial f(\theta,s)}{\partial s} = 0$$

$$= \frac{\partial}{\partial s} \ln|C| + \frac{\partial}{\partial s} \{\underline{\gamma}^T C^{-1} \underline{\gamma}\}$$

$$= \frac{\partial}{\partial s} \left\{ \ln f(s) + \frac{1}{f(s)} (b_1 s + b_0) \right\}$$

where $f(s) = a_2 s^2 + a_1 s + a_0$,

$$= \frac{f'(s)}{f(s)} - \frac{f'(s)}{f^2(s)} (b_1 s + b_0) + \frac{b_1}{f(s)}$$

$$= \frac{f'(s)f(s) - f'(s)(b_1 s + b_0) + b_1 f(s)}{f^2(s)}$$

At a stationary point,

$$f'(s)f(s) - f'(s)(b_1 s + b_0) + b_1 f(s) = 0 \qquad \text{(A.46)}$$

This is a third order equation in s, having either 3 real roots or one real and a pair of complex conjugate. Since $f'(s)$ is of degree one less than the degree of $f(s)$, (A.46) will be of odd degree in general, ensuring the existence of at least one real root.

In higher dimensions, the coefficients of (A.46) can be calculated iteratively and its real roots extracted by numerical methods. Let the real roots of (A.46) be $s_1, \ldots, s_r$, and let the roots for which

$$\left.\frac{\partial^2 f(\theta, s)}{\partial s^2}\right|_{s=s_i} > 0$$

be $s_1, \ldots, s_t$, $t \leq r$, after reordering.

The value that minimises $f(\theta,s)$ can then be obtained by simply comparing the values $f(\theta,s_i)$, $i=1,\ldots,t$, and choosing the s_i that yields the minimum $f(\theta,s_i)$.

I.5: *Numerical Algorithms Group (NAG) subroutines for the minimisation of scalar functions and the generation of pseudo-random numbers.*

E04ABF

1. Purpose

E04ABF searches for a minimum, in a given finite interval, of a continuous function of a single variable, using function values only. The method (based on quadratic interpolation) is intended for functions which have a continuous first derivative (although it will usually work if the derivative has occasional discontinuities).

IMPORTANT: before using this routine, read the appropriate machine implementation document to check the interpretation of italicised terms and other implementation-dependent details.

2. Specification (FORTRAN IV)

```
      SUBROUTINE E04ABF(FUNCT,E1,E2,A,B,MAXCAL,X,F,IFAIL)
C     INTEGER MAXCAL,IFAIL
C     real E1,E2,A,B,X,F
C     EXTERNAL FUNCT
```

3. Description

E04ABF is applicable to problems of the form:

Minimize $F(X)$

subject to $A \leq X \leq B$.

It normally computes a sequence of X values which tend in the limit to a minimum of F(X) subject to the given bounds. It also progressively reduces the interval [A, B] in which the minimum is known to lie. It uses the safeguarded quadratic-interpolation method described in [1].

The user must supply a subroutine FUNCT to evaluate F(X). The parameters E1 and E2 together specify the accuracy

$$Tol(X) = E1 \times |X| + E2$$

to which the position of the minimum is required. Note that FUNCT is never called at any point which is closer than Tol(X) to a previous point.

If the original interval [A, B] contains more than one minimum, E04ABF will normally find one of the minima.

4. References

[1] GILL, P.E. and MURRAY, W.
Safeguarded steplength algorithms for optimization using descent methods.
National Physical Laboratory Report NAC 37, 1973.

5. Parameters

FUNCT - SUBROUTINE.
This routine must be supplied by the user to calculate the value of the function F(X) at any point X in [A, B]. It should be tested separately before being used in conjunction with E04ABF. Its specification is:

```
      SUBROUTINE FUNCT(XC,FC)
      real XC,FC
```

XC - *real*.
XC specifies the point at which the function value is required. FUNCT must not change the value of XC.

FC - *real*.
FUNCT must set FC to the value of the function at the current point XC, i.e. set FC = F(XC).

Note: FUNCT must be declared as EXTERNAL in the (sub)program from which E04ABF is called.

E1 - *real*.
Before entry, E1 must be set to the relative accuracy to which the position of a minimum is required. (Note that, since E1 is a relative tolerance, the scaling of X is automatically taken into account.)

E1 should be no smaller than 2ε, and preferably not much less than $\sqrt{\varepsilon}$, where ε is the relative machine precision as given by NAG Library routine X02AAF.

If the user sets E1 to 0.0 (or to any value less than ε), E04ABF will reset E1 to the default value $\sqrt{\varepsilon}$ before starting the minimization process.

E2 - *real*.
Before entry, E2 must be set to the absolute accuracy to which the position of a minimum is required. E2 should be no smaller than 2ε. If the user sets E2 to 0.0 (or to any value less than ε), E04ABF will reset it to the default value $\sqrt{\varepsilon}$.

5. Parameters (contd)

A - *real*.
Before entry, A must be set to the lower bound of the interval containing a minimum. On exit, A contains an improved lower bound on the position of the minimum.

B - *real*.
Before entry, B must be set to the upper bound of the interval containing a minimum. On exit, B contains an improved upper bound on the position of the minimum.

MAXCAL - INTEGER.
Before entry, MAXCAL must be set to the maximum number of evaluations of F(X) which the user is prepared to allow. MAXCAL ≥ 3. (Few problems will require more than 30.) There will be an error exit (see Section 6) after MAXCAL calls of FUNCT.

On exit, MAXCAL contains the total number of times that FUNCT was actually called.

X - *real*.
On exit, X contains the estimated position of the minimum.

F - *real*.
On exit, F contains the value of the function at the final point in X.

IFAIL - INTEGER.
Before entry, IFAIL must be set to 0 or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details. Unless the routine detects an error (see Section 6), IFAIL contains 0 on exit.

For this routine, because the values of output parameters may be useful even if IFAIL ≠ 0 on exit, users are recommended to set IFAIL to 1 before entry. It is then essential to test the value of IFAIL on exit.

6. Error Indicators

Errors detected by the routine:-

IFAIL = 1 Parameter outside expected range. This failure will occur if, on entry, A + E2 ≥ B, MAXCAL < 3, IFAIL < 0 or IFAIL > 1.

6. Error Indicators (contd)

IFAIL = 2 The number of calls of FUNCT has exceeded MAXCAL. This may have happened simply because MAXCAL was set too small for a particular problem, or may be due to a mistake in the user's routine FUNCT. If no mistake can be found in FUNCT, restart E04ABF (preferably with values of A and B given on exit from the previous call of E04ABF).

7. Auxiliary Routines

This routine calls the NAG Library routines E04ABZ, P01AAF and X02AAF.

8. Timing

This depends on the behaviour of F(X), the accuracy demanded and the length of the interval [A, B]. Unless F(X) can be evaluated very quickly, the run time will usually be dominated by the time spent in FUNCT.

9. Storage

There are no internally declared arrays.

10. Accuracy

If F(X) is δ-unimodal (see Chapter Introduction) for some $\delta < \text{Tol}(X)$, where $\text{Tol}(X) = E1 \times |X| + E2$, then, on exit, X approximates the minimum of F(X) in the original interval [A, B] with an error less than $3 \times \text{Tol}(X)$.

11. Further Comments

If F(X) has more than one minimum in the original interval [A, B], E04ABF will determine an approximation X (and improved bounds A and B) for one of the minima.

If E04ABF finds an X such that $F(X - \Delta_1) > F(X) < F(X + \Delta_2)$ for some $\Delta_1, \Delta_2 \geq \text{Tol}(X)$, the interval $[X - \Delta_1, X + \Delta_2]$ will be regarded as containing a minimum, even if F(X) is less than $F(X - \Delta_1)$ and $F(X + \Delta_2)$ only due to rounding errors in the user-supplied routine. Therefore FUNCT should be programmed to calculate F(X) as accurately as possible, so that E04ABF will not be liable to find a spurious minimum.

G05ADF

1. Purpose

G05ADF returns a pseudo-random number from the standard normal distribution.

IMPORTANT: before using this routine, read the appropriate machine implementation document to check the interpretation of italicised terms and other implementation-dependent details.

2. Specification (FORTRAN IV)

```
      real FUNCTION G05ADF(X)
C     real X
```

3. Description

A sequence of pseudo-random numbers, z_i, from the standard normal distribution is generated by successive calls to G05ADF. Alternate elements of the sequence are determined from the equations

$$z_{2i-1} = \sqrt{-2\ ln\ (x_{2i-1})} \times \text{Sin}\ (2\pi x_{2i})$$

$$z_{2i} = \sqrt{-2\ ln\ (x_{2i-1})} \times \text{Cos}\ (2\pi x_{2i})$$

where x_i is a pseudo-random number generated by G05AAF.

4. References

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A note on the generation of random normal deviates.
Ann. Math. Stats. Vol. 29, pp. 610-611, 1958.

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The Art of Computer Programming, Vol. 2.
Addison-Wesley, 1969.

[3] NEAVE, H.
A Random Number Package.
Computer Applications in the Natural and Social Sciences, No. 14,
Dept. of Geography, University of Nottingham, 1972.

[4] WEATHERBURN, C.E.
A First Course in Mathematical Statistics,
Cambridge University Press, 1949.

5. Parameters

X - *real*.
X is a dummy parameter, required by the FORTRAN compiler.

6. Error Indicators None.

7. Auxiliary Routines

This routine calls the NAG Library routine G05AAF.

8. Timing

The timing will vary between machine ranges. The appropriate machine implementation document may give specific details.

9. Storage

There are no internally declared arrays.

10. Accuracy

For a given starting value the sequence of pseudo-random numbers will repeat itself after a finite number of direct or indirect calls to G05AAF. The relevant machine implementation document may give details. Each call of G05ADF alternately calls G05AAF twice and then zero times.

11. Further Comments

A different starting point in the sequence of numbers generated may be obtained by using the NAG Library routines G05BAF or G05BBF. The sequence is also affected by direct or indirect calls to G05AAF elsewhere in the program.

G05AAF

1. Purpose

G05AAF returns a pseudo-random number from a uniform (rectangular) distribution on the range (0,1).

IMPORTANT: before using this routine, read the appropriate machine implementation document to check the interpretation of italicised terms and other implementation-dependent details.

2. Specification (FORTRAN IV)

```
      real FUNCTION G05AAF(X)
C     real X
```

3. Description

Two multiplicative congruential sequences

$$x_{1,r+1} = b_1 x_{1,r} \pmod{M}$$

and

$$x_{2,r+1} = b_2 x_{2,r} \pmod{M}$$

are generated. A sequence of pseudo-random numbers, x_{r+1}, is then formed using

$$x_{r+1} = x_{1,r+1} + x_{2,r+1} \pmod{M}$$

and these are scaled by M to produce the required sequence. The values of the constants used are $M = 2^{46}$, $b_1 = 3^{15}$, $b_2 = 5^9$, $x_{1,0,} = x_{2,0,} = 1234567$.

4. References

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Dept. of Geography, University of Nottingham, 1972.

[3] WEATHERBURN, C.E.
A First Course in Mathematical Statistics.
Cambridge University Press, 1949.

5. Parameters

X - *real*.
X is a dummy parameter, required by the FORTRAN compiler.

6. Error Indicators None.

7. Auxiliary Routines None.

8. Timing

The timing will vary between machine ranges. The appropriate machine implementation document may give specific details.

9. Storage

This may vary between machine ranges but should be small (less than 20 *real* elements).

10. Accuracy

For a given starting value the sequence of pseudo-random numbers will repeat itself after a finite number of calls of the routine. The relevant machine implementation document may give details.

11. Further Comments

The routine is written in machine code. It may be initialised to a different starting point in the sequence by using the NAG Library routines G05BAF or G05BBF.

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