Nonsmooth and Nonconvex Optimization for the Design and Order Reduction of Robust Controllers Used in Smart Structures

Chapter · January 2013

DOI: 10.1007/978-94-007-6573-3_26

3 authors, including:

Anastasios D. Pouliezos
Technical University of Crete

Georgios Eleftherios Stavroulakis
Technical University of Crete

Nonsmooth and Nonconvex Optimization for the Design and Order Reduction of Robust Controllers Used in Smart Structures

A.J. Moutsopoulou, A.T. Pouliezos, and G.E. Stavroulakis

Abstract H-infinity controller design for linear systems is a difficult, nonconvex typically nonsmooth optimization problem when the controller is fixed to be of order less than the one of the open-loop plant, a requirement with some importance in embedded smart systems. In this paper we use a new Matlab package called HIFOO, aimed at solving fixed-order stabilization and local optimization problems; it is based on a new hybrid solution algorithm. The problem is to reduce the vibration of the smart system using H-infinity control and nonsmooth optimization.

Keywords Fixed-order controller design · H-infinity control · Nonconvex optimization · Nonsmooth optimization · Smart structures · Robust structural control

1 Introduction

A composite beam laminated with piezoelectric sensors and actuators and subjected to external loads is considered as a model example in this paper. The governing equation of the beam is formulated. Finite elements are used to approximate the dynamic response of the beam. Vibration control problem for flexible structure is considered and performance specification is stated in terms of a disturbance attenuation requirement for external disturbances. H-infinity (H_∞ control strategy) is applied to

A.J. Moutsopoulou (✉)
Department of Civil Engineering, Technological Educational Institute of Crete, 71004 Heraklion, Greece
e-mail: amalia@staff.teicrete.gr

A.T. Pouliezos · G.E. Stavroulakis
Department of Production Engineering and Management, Technical University of Crete, 73100 Chania, Greece

A.T. Pouliezos
e-mail: tasos@dpem.tuc.gr

G.E. Stavroulakis
e-mail: gestavr@dpem.tuc.gr

M. Papadrakakis et al. (eds.), *Computational Methods in Earthquake Engineering*, Computational Methods in Applied Sciences 30, DOI 10.1007/978-94-007-6573-3_26,

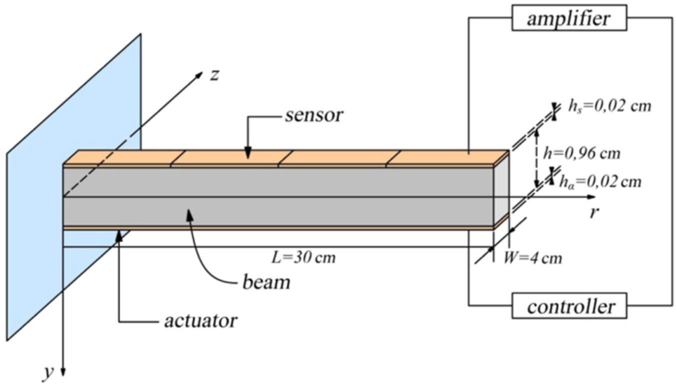


Fig. 1 Smart beam

solve the posed problem. Considering the uncertainty, which arises from neglecting higher order, dynamics, insufficient knowledge of the real plant parameters, external disturbances and measure noise, H_∞ robust controller is designed [1–5].

By using the classical approach a H_∞ controller of order equal to 36 has been found. The fact that controller order, which is equal to the order of the system, is relatively higher than the order of classical controllers such as PI and LQR has led a number of researchers to develop order reduction algorithms. The most widely used such algorithm, known as HIFOO, has been implemented in a Matlab environment, and is the one used in the following procedure [6–10].

HIFOO is a public-domain Matlab package initially designed for H_∞ fixed-order controller synthesis, using nonsmooth nonconvex optimization techniques. It was later on extended to multi-objective synthesis, including strong and simultaneous stabilization under H_∞ constraints. HIFOO relies upon HANSO, a general purpose implementation of an hybrid algorithm for nonsmooth optimization, mixing standard quasi-Newton and gradient sampling techniques. The acronym HIFOO stands for H_∞ fixed order optimization, and the package is aimed at designing a stabilizing linear controller of fixed order for a linear plant in standard state-space configuration while minimizing the H_∞ norm of the closed-loop transfer function.

2 Mathematical Modelling

A cantilever slender beam with rectangular cross-sections is considered. Four pairs of piezoelectric patches are embedded symmetrically at the top and the bottom surfaces of the beam, as shown in Fig. 1.

The beam is from graphite-epoxy T300-976 and the piezoelectric patches are PZT G1195N [5, 11]. The top patches act like sensors and the bottom like actuators. The resulting composite beam is modelled by means of the classical laminated technical theory of bending. Let us assume that the mechanical properties of both

Table 1 Parameters of the composite beam

Parameters	Values
Beam length, L	0.3 m
Beam width, W	0.04 m
Beam thickness, h	0.0096 m
Beam density, ρ	1600 kg/m^3
Young's modulus of the beam, E	1.5×10^{11} N/m^2
Piezoelectric constant, d_{31}	254×10^{-12} m/V
Electric constant, ξ_{33}	11.5×10^{-3} V m/N
Young's modulus of the piezoelectric element	1.5×10^{11} N/m^2
Width of the piezoelectric element	$b_S = b_A = 0.04$ m
Thickness of the piezoelectric element	$h_S = h_A = 0.0002$ m

the piezoelectric material and the host beam are independent in time. The thermal effects are considered to be negligible as well [12, 13].

The beam has length L, width W and thickness h. The sensors and the actuators have width b_S and b_A and thickness h_S and h_A, respectively. The electromechanical parameters of the beam of interest are given in Table 1.

2.1 Reduced Model of the Piezoelectric Composite

In order to derive the basic equations for piezoelectric sensors and actuators (S/As), we assume that:

- The piezoelectric S/A are bonded perfectly on the host beam;
- The piezoelectric layers are much thinner then the host beam;
- The piezoelectric material is homogeneous, transversely isotropic and linearly elastic;
- The piezoelectric S/A are transversely polarized (in the z-direction) [5, 13].

Under these assumptions the three-dimensional linear constitutive equations are given by [1, 9],

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{55} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xz} \end{Bmatrix} - \begin{bmatrix} d_{31} \\ 0 \end{bmatrix} E_z \right) \tag{2.1}$$

$$D_z = Q_{11} d_{31} \varepsilon_{xx} + \xi_{xx} E_z \tag{2.2}$$

where σ_{xx}, σ_{xz} denote the axial and shear stress components, D_z, denotes the transverse electrical displacement; $\varepsilon_{xx}$and ε_{xz} are a axial and shear strain components; Q_{11}, and Q_{55}, denote elastic constants; d_{31}, and ξ_{33}, denote piezoelectric and dielectric constants, respectively. Equation (2.1) describes the inverse piezoelectric effect and Eq. (2.2) describes the direct piezoelectric effect. E_z, is the transverse component of the electric field that is assumed to be constant for the piezoelectric layers and its components in xy-plain are supposed to vanish. If no electric field is applied in the sensor layer, the direct piezoelectric equation (2.2) gets the form,

$$D_z = Q_{11} d_{31} \varepsilon_{xx} \tag{2.3}$$

and it is used to calculate the output charge created by the strains in the beam [2].

2.2 Equations of Motion

The length, width and thickness of the host beam are denoted by L, b and h. The thickness of the sensor and actuator is denoted by h_S and h_A. We assume that:

- The beam centroidal and elastic axis coincides with the x-coordinate axis so that no bending-torsion coupling is considered;
- The axial vibration of the host beam centreline is considered negligible;
- The displacement field $\{u\} = (u_1, u_2, u_3)$ is obtained based on the usual Timoshenko assumptions [14],

$$\begin{aligned} u_1(x, y, z) &\approx z\varphi(x, t) \\ u_2(x, y, z) &\approx 0 \\ u_3(x, y, z) &\approx \omega(x, t) \end{aligned} \tag{2.4}$$

  where φ is the rotation of the beams cross-section about the positive y-axis and w is the transverse displacement of a point of the centroidal axis ($y = z = 0$).

The strain displacement relations can be applied to Eq. (2.4) to give,

$$\varepsilon_{xx} = z\frac{\partial \varphi}{\partial x}, \qquad \varepsilon_{xz} = \frac{\partial \omega}{\partial x} \tag{2.5}$$

We suppose that the transverse shear deformation ε_{xx} is equal to zero [14].

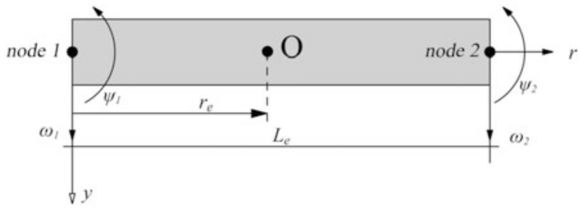
In order to derive the equations of the motion of the beam we use Hamilton's principle,

$$\int_{t_2}^{t_1} (\delta T - \delta U + \delta W) dt = 0 \tag{2.6}$$

where T [15] is the total kinetic energy of the system, U is the potential (strain) energy and W is the virtual work done by the external mechanical and electrical loads and moments. The first variation of the kinetic energy is given by,

$$\begin{aligned} \delta T &= \frac{1}{2}\int_V \rho \left\{\frac{\partial u}{\partial t}\right\}^r \left\{\frac{\partial u}{\partial t}\right\} dV \\ &= \frac{b}{2}\int_0^L \int_{-\frac{h}{2}-h_A}^{\frac{h}{2}+h_s} \rho \left(z\frac{\partial \varphi}{\partial t}\delta\frac{\partial \varphi}{\partial t} + \frac{\partial \omega}{\partial t}\delta\frac{\partial \omega}{\partial t} \right) dz dx \end{aligned} \tag{2.7}$$

Fig. 2 Beam finite element



The first variation of the potential energy is given by,

$$\delta U = \frac{1}{2}\int_V \delta\{\varepsilon\}^T\{\sigma\}dV$$
$$= \frac{b}{2}\int_0^L \int_{-\frac{h}{2}-h_A}^{\frac{h}{2}+h_s} \left[Q_{11}\left(z\frac{\partial\omega}{\partial x}\delta\right)\left(z\frac{\partial\omega}{\partial x}\right)\right]dzdx \tag{2.8}$$

If the load consists only of moments induced by piezoelectric actuators and since the structure has no bending twisting couple then the first variation of the work has the form [15],

$$\delta W = b\int_0^L M^A\delta\left(\frac{\partial\varphi}{\partial x}\right)dx \tag{2.9}$$

where M^A is the moment per unit length induced by the actuator layer and is given by,

$$M^A = \int_{-\frac{h}{2}-h_A}^{-\frac{h}{2}} z\sigma_{xx}^A dz = \int_{-\frac{h}{2}-h_A}^{-\frac{h}{2}} zQ_{11}d_{31}E_z^A dz\left(E_z^A = \frac{V_A}{h_A}\right) \tag{2.10}$$

2.3 Finite Element Formulation

We consider a beam element of length L_e, which has two mechanical degrees of freedom at each node: one translational ω_1 (respectively ω_2) in direction z and one rotational φ_1 (respectively φ_2), as it is shown in Fig. 2. The vector of nodal displacements and rotations q_e is defined as [2],

$$q_e^r = [\omega_1, \psi_1, \omega_2, \psi_2] \tag{2.11}$$

The beam element transverse deflection $\omega(x,t)$ and the beam element rotation $\psi(x,t)$ of the beam are continuous and they are interpolated within the beam by using Hermitian linear shape functions H_i^ω and H_i^ψ as follows [4],

$$\omega(x,t) = \sum_{i=1}^{4} H_i^\omega(x)q_i(t)$$
$$\psi(x,t) = \sum_{i=1}^{4} H_i^\psi(x)q_i(t) \tag{2.12}$$

This classical finite element procedure leads to the approximate (discretized) variational problem. For a finite element the discrete differential equations are obtained by substituting the discretized expressions (2.12) into Eqs. (2.7) and (2.8) to evaluate the kinetic and strain energies. Integrating over spatial domains and using the Hamilton principle (2.6) the equation of motion for a beam element are expressed in terms of nodal variable q as follows [3, 5, 11],

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = f_m(t) + f_e(t) \tag{2.13}$$

where M is the generalized mass matrix, D the viscous damping matrix, K the generalized stiffness matrix, f_m the external loading vector and f_e the generalized control force vector produced by electromechanical coupling effects. The independent variable $q(t)$ is composed of transversal deflections ω_1 and rotations ψ_1, i.e., [16]

$$q(t) = \begin{bmatrix} \omega_1 \\ \psi_1 \\ \vdots \\ \omega_n \\ \psi_n \end{bmatrix} \tag{2.14}$$

where n is the number of nodes used in analysis. Vectors ω and f_m are positive upwards. To transform to state-space control representation, let (in the usual manner),

$$\dot{x}(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \tag{2.15}$$

Furthermore to express $f_e(t)$ as $Bu(t)$ we write it as $f_e^* u$ where f_e^* the piezoelectric force is for a unit applied on the corresponding actuator, and u represents the voltages on the actuators. Furthermore, $d(t) = f_m(t)$ is the disturbance vector [17]. Then,

$$\dot{x}(t) = \begin{bmatrix} O_{2n\times 2n} & I_{2n\times 2n} \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x(t) + \begin{bmatrix} O_{2n\times 2n} \\ M^{-1}f_e^* \end{bmatrix} u(t) + \begin{bmatrix} O_{2n\times 2n} \\ M^{-1} \end{bmatrix} \tag{2.16}$$

$$= Ax(t) + Bu(t) + Gd(t) = Ax(t) + \begin{bmatrix} B & G \end{bmatrix} \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}$$

$$= Ax(t) + \tilde{B}\tilde{u}(t) \tag{2.17}$$

The previous description of the dynamical system will be augmented with the output equation (displacements only measured) [2],

$$y(t) = \begin{bmatrix} x_1(t) & x_3(t) & \cdots & x_{n-1}(t) \end{bmatrix}^T = Cx(t) \tag{2.18}$$

In this formulation u is $n \times 1$ (at most, but can be smaller), while d is $2n \times 1$. The units used are compatible for instance m, rad, sec and N.

3 Design Objectives and System Specifications

For the description of system uncertainties the known from the control theory Δ technique is used. With this technique, which will be outlined later, the uncertainties are introduced in the system as additional feedback branches, similarly to the introduction of control branches. The robust control design tries then to design the optimal control feedback under perturbation coming from the worst uncertainty feedback (for details [12]).

The structured singular value of a transfer function matrix is defined as,

$$\mu(M) = \begin{cases} \dfrac{1}{\min_{k_m}\{\det(I - k_m M\Delta) = 0, \bar{\sigma}(\Delta) \leq 1\}} \\ 0, \quad \text{if no such quantity is defined} \end{cases} \tag{3.1}$$

In words it defines the smallest structured Δ (measured in terms of $\bar{\sigma}(\Delta)$) which makes $\det(I - M\Delta) = 0$: then $\mu(M) = 1/\bar{\sigma}(\Delta)$. It follows that values of μ smaller than 1 are desired (the smaller the better: a larger variation is allowed) [12].

3.1 Design Objectives

Design objectives fall into two categories:

Nominal performance

1. Stability of closed loop system (plant + controller).
2. Disturbance attenuation with satisfactory transient characteristics (overshoot, settling time).
3. Small control effort.

Robust performance

4. (1)–(3) above should be satisfied in the face of modelling errors [12].

3.2 System Specifications

To obtain the required system specifications to meet the above objectives we need to represent our system in the so-called (N, Δ) structure. To do this let us start with the simple typical diagram of Fig. 3. In this diagram there are two inputs, d (disturbances) and n (noise), and two outputs, u (control vector) and x (state vector). In what follows it is assumed that,

$$\left\| \begin{matrix} d \\ n \end{matrix} \right\|_2 \leq 1, \qquad \left\| \begin{matrix} u \\ x \end{matrix} \right\|_2 \leq 1 \tag{3.2}$$

If that's not the case, appropriate frequency-dependent weights can transform original signals so that the transformed signals have this property.

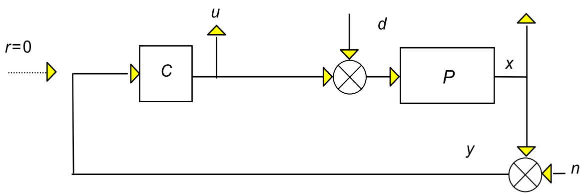


Fig. 3 Classical control block diagram

Redrawing Fig. 3 in frequency domain we get Fig. 4:

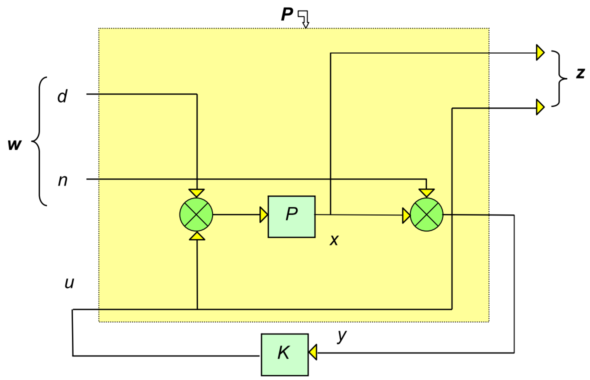


Fig. 4 Detailed two-port diagram

or in less detail Fig. 5,

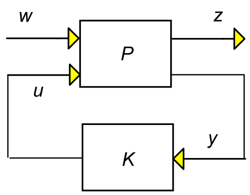


Fig. 5 Two-port diagram

with,

$$z = \begin{bmatrix} u \\ x \end{bmatrix}, \qquad w = \begin{bmatrix} d \\ n \end{bmatrix} \tag{3.3}$$

where z are the output variables to be controlled, and w the exogenous inputs.

Given that P has two inputs and two outputs it is, as usual, naturally partitioned as,

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{zw}(s) & P_{zu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \overset{op}{=} P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \tag{3.4}$$

Furthermore,

$$u(s) = K(s)y(s) \tag{3.5}$$

Substituting (3.5) in (3.4) gives the closed loop transfer function $N_{zw}(s)$,

$$N_{zw}(s) = P_{zw}(s) + P_{zu}(s)K(s)\big(I - P_{yu}(s)K(s)\big)^{-1} P_{yw}(s) \tag{3.6}$$

4 Controller Synthesis

It is possible to approximately synthesize a controller that achieves given performance in terms of the structured singular value μ.

In this procedure known as $(D, G\text{-}K)$ iteration (2.14) the problem of finding a μ-optimal controller K such that $\mu(\mathcal{F}_u(F(j\omega), K(j\omega))) \leq \beta$, $\forall\omega$, is transformed into the problem of finding transfer function matrices $D(\omega) \in \mathcal{D}$ and $G(\omega) \in \mathcal{G}$, such that,

$$\sup_{\omega} \bar{\sigma}\left[\left(\frac{D(\omega)\mathcal{F}_u(F(j\omega), K(j\omega))D^{-1}(\omega)}{\gamma} - jG(\omega)\right)\left(I + G^2(\omega)\right)^{-\frac{1}{2}}\right] \leq 1, \quad \forall\omega \tag{4.1}$$

Unfortunately this method does not guarantee even finding local maxima. However for complex perturbations a method known as D-K iteration is available (also implemented in Matlab) [11, 12]. It combines H_∞ synthesis and μ-analysis and often yields good results. The starting point is an upper bound on μ in terms of the scaled singular value,

$$\mu(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}\left(DND^{-1}\right) \tag{4.2}$$

The idea is to find the controller that minimizes the peak over frequency of its upper bound, namely,

$$\min_{K}\left(\min_{D \in \mathcal{D}} \|DN(K)D^{-1}\|_\infty\right) \tag{4.3}$$

by alternating between minimizing $\|DN(K)D^{-1}\|_\infty$ with respect to either K or D (while holding the other fixed) [12, 18, 19].

1. *K-step*. Synthesize an $\mathcal{H}_\infty$ controller for the scaled problem $\min_K \|DN(K) \times D^{-1}\|_\infty$ with fixed $D(s)$.
2. *D-step*. Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed N.
3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase transfer function $D(s)$ and go to Step 1.

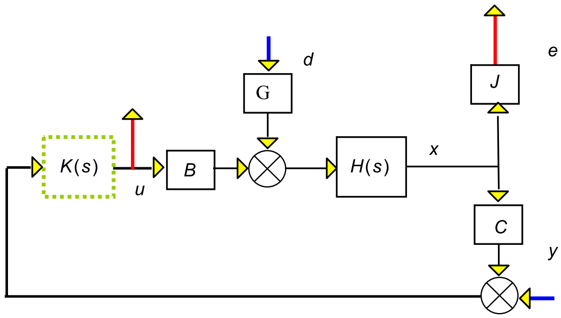
5 Relation to the Beam Control Problem

Our aim is to find the appropriate N matrix, as defined in (3.6). To this aim it is useful, to derive the input-output relations for the original model,

$$\begin{bmatrix} u \\ e \end{bmatrix} = F(s)\begin{bmatrix} d \\ n \end{bmatrix} \quad \Rightarrow \quad z = F(s)w \tag{5.1}$$

as depicted in Fig. 6, where the beam is described by,

Fig. 6 Beam with controller



$$\dot{\mathbf{x}}(t) = Ax(t) + [\,B \quad G\,]\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{d}(t)\end{bmatrix} \tag{5.2}$$

$$\Rightarrow \quad H(s) = (sI - A)^{-1} \tag{5.3}$$

and J is used to pick up those states that we are interested in regulating (may be different from y). In most of the displacements at the four nodes of the discretized structure are assumed to be measured i.e. J will be,

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4}$$

To continue, use appropriate weighting and redraw Fig. 6 to fit our problem:

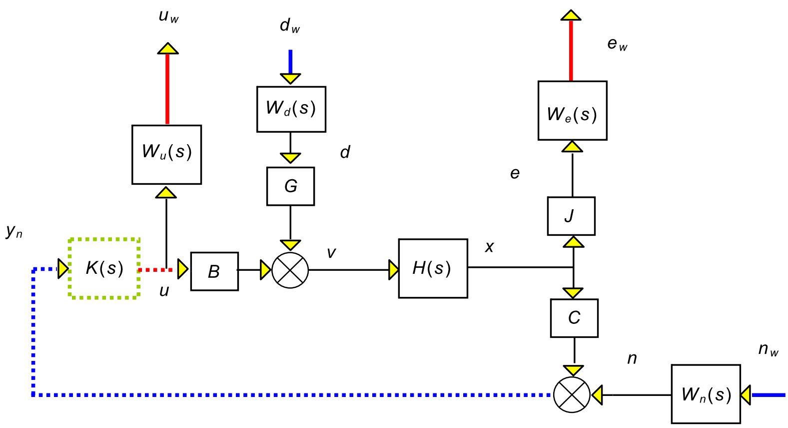
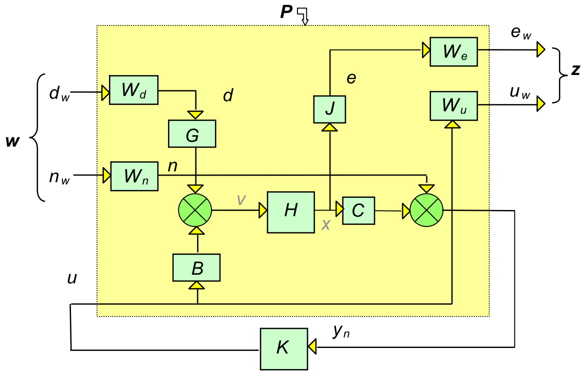


Fig. 7 Weighted block diagram for the beam problem

Then redraw Fig. 7 as a two port diagram, similar to Fig. 8:

Fig. 8 Two port diagram for the beam problem



In Fig. 8 x, v are auxiliary signals.We are looking for,

$$Q_{zw}(s) = P_{zw}(s) + P_{zu}(s)K(s)\big(I - P_{yu}(s)K(s)\big)^{-1} P_{yw}(s) \tag{5.5}$$

such that,

$$z = Q_{zw}w = F_1(P, K)w \tag{5.6}$$

We need to find $P(s)$. The necessary transfer functions are,

$$\begin{aligned} e_w &= W_e Jx = W_e JHv = W_e JH(GW_d d_w + Bu) \\ &= W_e JHGW_d d_w + W_e JHBu \end{aligned} \tag{5.7}$$

$$u_w = W_u u \tag{5.8}$$

$$\begin{aligned} y_n &= Cx + W_n n_w = CHv + W_n n_w = CH(GW_d d_w + Bu) + W_n n_w \\ &= CHGW_d d_w + CHBu + W_n n_w \end{aligned} \tag{5.9}$$

Combining all these gives,

$$\begin{bmatrix} u_w \\ e_w \\ \cdots \\ y_n \end{bmatrix} = \left[\begin{array}{cc:c} 0 & 0 & W_u \\ W_e JHGW_d & 0 & W_e JHB \\ \hdashline CHGW_d & W_n & CHB \end{array}\right] \begin{bmatrix} d_w \\ n_w \\ \cdots \\ u \end{bmatrix} \tag{5.10}$$

or,

$$\begin{bmatrix} z \\ y_n \end{bmatrix} = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{5.11}$$

where,

$$\begin{aligned} P_{zw} &= \begin{bmatrix} 0 & 0 \\ W_e JHGW_d & 0 \end{bmatrix}, \qquad P_{zu} = \begin{bmatrix} W_u \\ W_e JHB \end{bmatrix} \\ P_{yw} &= \begin{bmatrix} CHGW_d & W_n \end{bmatrix}, \qquad P_{yu} = CHB \end{aligned} \tag{5.12}$$

One more step is needed, however to get the Q_{ij}'s. We do this using

$$\begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} (I - KCHB)^{-1}KCHG & (I - KCHB)^{-1}K \\ J(I - HBKC)^{-1}HG & J(I - HBKC)^{-1}HBK \end{bmatrix} \begin{bmatrix} d \\ n \end{bmatrix} \tag{5.13}$$

and noting that,

$$d = W_d d_w, \qquad n = W_n n_w, \qquad e_w = W_e e, \qquad u_w = W_u u$$

$$\begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} W_u^{-1} u_w \\ W_e^{-1} e_w \end{bmatrix} = F(s) \begin{bmatrix} d \\ n \end{bmatrix} = F(s) \begin{bmatrix} W_d d_w \\ W_n n_w \end{bmatrix} \quad \Rightarrow$$

$$\begin{bmatrix} u_w \\ e_w \end{bmatrix} = \begin{bmatrix} W_u & \\ & W_e \end{bmatrix} F(s) \begin{bmatrix} W_d & \\ & W_n \end{bmatrix} \begin{bmatrix} d_w \\ n_w \end{bmatrix} \tag{5.14}$$

or,

$$\begin{bmatrix} u_w \\ e_w \end{bmatrix} = \begin{bmatrix} W_u(I - KCHB)^{-1} KCHGW_d & W_u(I - KCHB)^{-1} KW_n \\ W_e J(I - HBKC)^{-1} HGW_d & W_e J(I - HBKC)^{-1} HBKW_n \end{bmatrix} \times \begin{bmatrix} d_w \\ n_w \end{bmatrix} \tag{5.15}$$

Let us insert the previous matrices in,

$$z = Q_{zw} w \quad \text{or} \quad \begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} d \\ n \end{bmatrix} \tag{5.16}$$

To express P in state space, form the natural partitioning,

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} P_{zw}(s) & P_{zu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{bmatrix} \tag{5.17}$$

(where the packed form has been used), while the corresponding form for K is,

$$K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{5.18}$$

Equation (5.18) defines the equations,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \\ \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \end{aligned} \tag{5.19}$$

and,

$$\dot{x}_K(t) = A_K x_K(t) + B_K y(t) \tag{5.20}$$

$$u(t) = C_K x_K(t) + D_K y(t) \tag{5.21}$$

To find the matrices involved, we break the feedback loop and use the relevant equations, see Fig. 9:

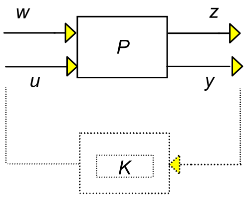


Fig. 9 Open loop structure

To get the structure in state space form, relate the inputs, outputs, states and input/output to the controller:

$$\begin{aligned}
\dot{x}_F &= Ax_F + (Gd + Bu), & x &= Ix_F \\
\dot{x}_u &= A_u x_u + B_u u, & u_w &= C_u x_u + D_u u \\
\dot{x}_e &= A_e x_e + B_e Jx, & e_w &= C_e x_e + D_e Jx \\
\dot{x}_{nw} &= A_{nw} x_{nw} + B_{nw} n_w, & n &= C_{nw} x_{nw} + D_{nw} n_w \\
\dot{x}_{dw} &= A_{dw} x_{dw} + B_{dw} d_w, & d &= C_{dw} x_{dw} + D_{dw} d_w \\
& & y &= Cx + n
\end{aligned} \tag{5.22}$$

Let,

$$x = \begin{bmatrix} x_F \\ x_u \\ x_e \\ x_{nw} \\ x_{dw} \end{bmatrix}, \qquad y = y, \qquad w = \begin{bmatrix} d_w \\ n_w \end{bmatrix}, \qquad z = \begin{bmatrix} u_w \\ e_w \end{bmatrix}, \qquad u = u \tag{5.23}$$

Substituting the internal signals d, n, e and x from (5.22), yields,

$$\begin{aligned}
\begin{bmatrix} \dot{x}_F \\ \dot{x}_u \\ \dot{x}_e \\ \dot{x}_{nw} \\ \dot{x}_{dw} \end{bmatrix} &= \begin{bmatrix} A & 0 & 0 & 0 & GC_{dw} \\ 0 & A_u & 0 & 0 & 0 \\ B_e J & 0 & A_e & 0 & 0 \\ 0 & 0 & 0 & A_{nw} & 0 \\ 0 & 0 & 0 & 0 & A_{dw} \end{bmatrix} \begin{bmatrix} x_F \\ x_u \\ x_e \\ x_{nw} \\ x_{dw} \end{bmatrix} \\
&\quad + \begin{bmatrix} GD_{dw} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_{nw} \\ B_{dw} & 0 \end{bmatrix} \begin{bmatrix} d_w \\ n_w \end{bmatrix} + \begin{bmatrix} B \\ B_u \\ 0 \\ 0 \\ 0 \end{bmatrix} u
\end{aligned} \tag{5.24}$$

$$\begin{bmatrix} u_w \\ e_w \end{bmatrix} = \begin{bmatrix} 0 & C_u & 0 & 0 & 0 \\ D_e J & 0 & C_e & 0 & 0 \end{bmatrix} \begin{bmatrix} x_F \\ x_u \\ x_e \\ x_{nw} \\ x_{dw} \end{bmatrix} + 0 \begin{bmatrix} d_w \\ n_w \end{bmatrix} + \begin{bmatrix} D_u \\ 0 \end{bmatrix} u \tag{5.25}$$

$$y = \begin{bmatrix} C & 0 & 0 & C_{nw} & 0 \end{bmatrix} \begin{bmatrix} x_F \\ x_u \\ x_e \\ x_{nw} \\ x_{dw} \end{bmatrix} + \begin{bmatrix} 0 & D_{nw} \end{bmatrix} \begin{bmatrix} d_w \\ n_w \end{bmatrix} + 0u \tag{5.26}$$

Therefore the matrices are:

$$A_1 = \begin{bmatrix} A & 0 & 0 & 0 & GC_{dw} \\ 0 & A_u & 0 & 0 & 0 \\ B_e J & 0 & A_e & 0 & 0 \\ 0 & 0 & 0 & A_{nw} & 0 \\ 0 & 0 & 0 & 0 & A_{dw} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} GD_{dw} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_{nw} \\ B_{dw} & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} B \\ B_u \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.27}$$

$$C_1 = \begin{bmatrix} 0 & C_u & 0 & 0 & 0 \\ D_e J & 0 & C_e & 0 & 0 \end{bmatrix}, \qquad D_{11} = 0, \qquad D_{12} = \begin{bmatrix} D_u \\ 0 \end{bmatrix} \tag{5.28}$$

$$C_2 = \begin{bmatrix} C & 0 & 0 & C_{nw} & 0 \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} 0 & D_{nw} \end{bmatrix}, \qquad D_{22} = 0 \tag{5.29}$$

As can be seen, in this configuration the size of the state vector is $16 + 4 + 4 + 4 + 8 = 36$ (16 is the size of the state vector, 4 is the size of the disturbance and of the noise and 8 is the size of the errors). This (36) will also be the size of the controller model $K(s)$. This number will be decreased, if some weight matrices are constant, by their corresponding order.

6 Results for H_∞ Control

In the simulations that follow the disturbance is the first mechanical load, i.e. 10 N at the free end of the beam,

$$f_m(t) = 10\text{ N} \tag{6.1}$$

Figure 10 shows the displacement time history at all nodal points of the beam, with and without control. Results are satisfactory, as recovery time is about 0.005 sec. We observe vibration reduction of 95 %.

Figure 11 displays the angle of rotation time history at all beam nodal points, with and without control, using H_∞, angle of rotation tends to vanish completely.

In Fig. 12 actuator voltage is plotted for all beam nodal points. Voltage lies below 500 V, witch is the piezoelectric limit.

In the simulations that follow the disturbance is the sinusoidal load

$$f_m(t) = q_0(t) \tag{6.2}$$

$$q_0(t) = 10\sin(0.01t) \tag{6.3}$$

In Fig. 13 displacement time history is presented for all four nodes of the beam, with and without control. We observe vibration reduction of 90 %. By employing the H_∞ control, vibration reduction is achieved; while the voltage applied is significantly lower that 500 V, which is the piezoelectric limit (Fig. 14).

Figure 15 shows the variation of angle of rotation for the four beam nodes, with and without control. Using H_∞, angle of rotation tends to vanish completely.

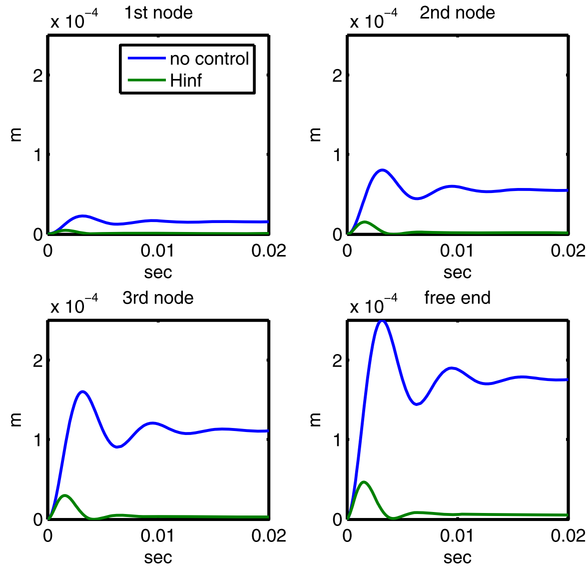


Fig. 10 Displacement at beam nodal points, with and without control, for the first mechanical loading

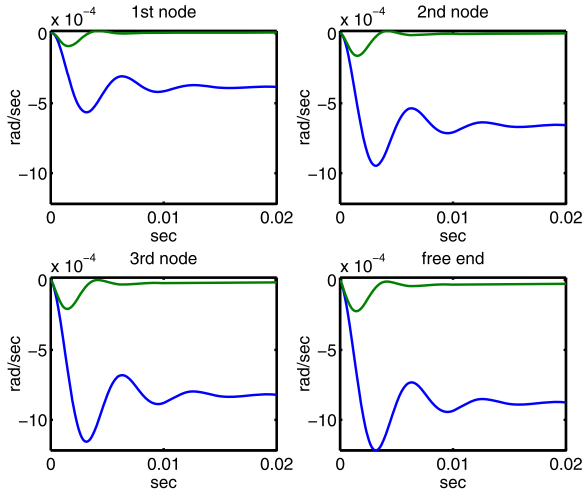


Fig. 11 Angle of rotation at beam nodal points, with and without control, for the first mechanical loading

7 Order Reduction of Controller H_∞

The H_∞ controller found is order 36. The fact that controller order, which is equal to the order of the system, is relatively higher than the order of classical controllers such as PI and LQR has led a number of researchers to develop order reduction algorithms. The most widely used such algorithm, known as HIFOO, has been implemented in a Matlab environment, and is the one used in the following procedure [8, 9].

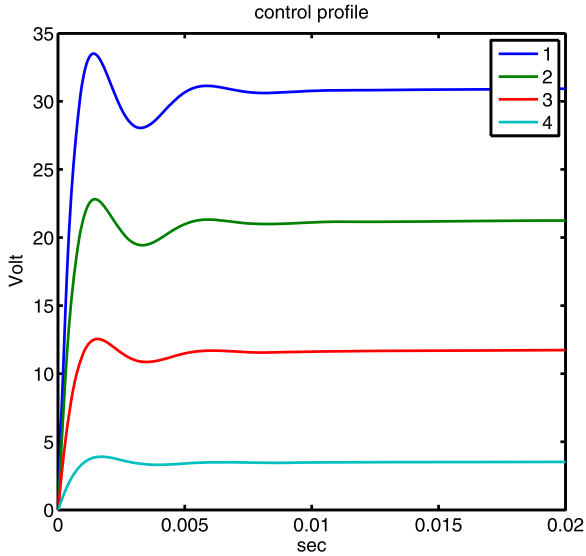


Fig. 12 Control produced voltage at beam nodal points

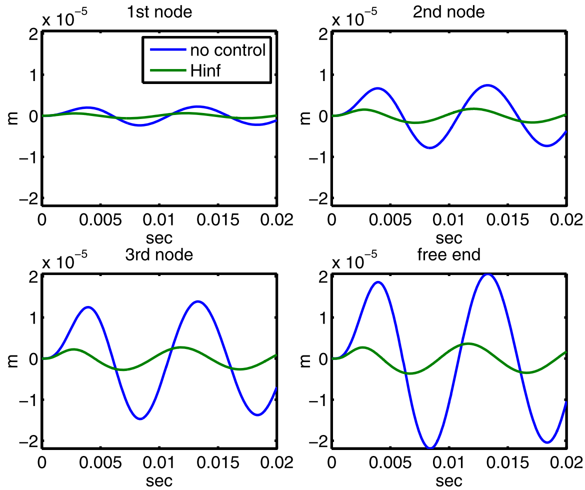


Fig. 13 Displacement at beam nodal points, with and without control, for the second mechanical loading

The general problem is to compute a controller of reduced order $n < 36$ while retaining the performance of the H_∞ criterion as well as the behaviour of a full order controller for the given system [7, 8].

As already mentioned that, the state space equations of our system are

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t)\\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t)\\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t)\end{aligned} \tag{7.1}$$

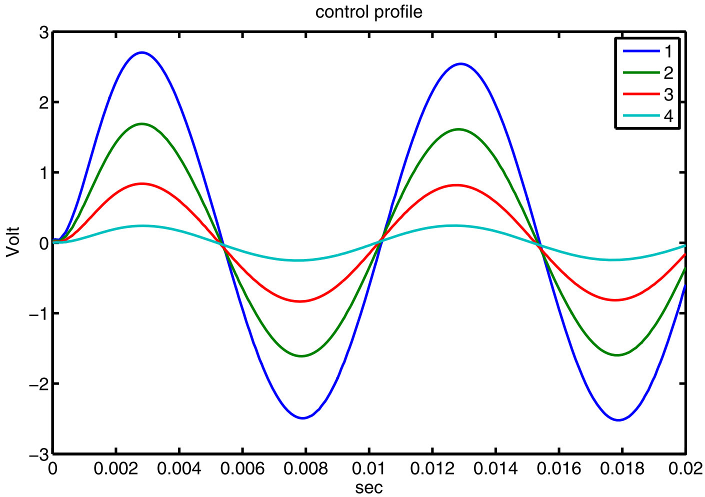
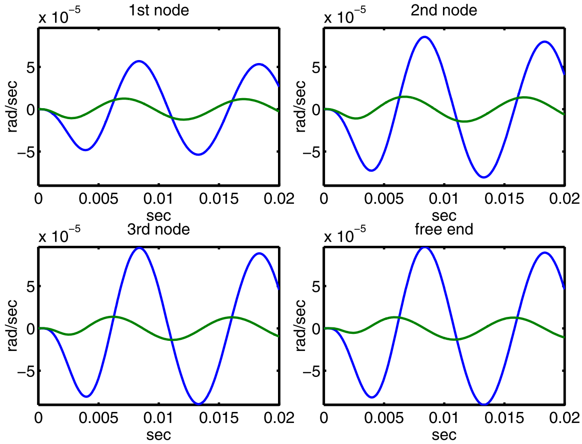


Fig. 14 Control produced voltage at beam nodal points, for the second mechanical loading

Fig. 15 Angle of rotation at beam nodal points, with and without control, for the second mechanical loading



and the state space equations for the controller K are

$$\begin{aligned} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\ u(t) &= C_K x_K(t) + D_K y(t) \end{aligned} \tag{7.2}$$

Let $\alpha(X)$ be the spectral abscissa of a matrix X that is the maximum real part of its eigenvalues. Then, we require not only that $\alpha(A_{CL}) < 0$, where A_{CL} is the closed-loop system matrix, but that $\alpha(A_k) < 0$ as well. The feasible set of A_k, that is the set of stable matrices, is not a convex set and has a boundary that is not smooth [6, 10].

The HIFOO procedure has two phases: stability and performance optimization [9, 20]. In the stability phase, HIFOO attempts to minimize

$$\max\big(\alpha\big(A_{CL}, \varepsilon\alpha(A_{CL})\big)\big) \tag{7.3}$$

where ε is a positive parameter that will be described shortly, until a controller is found for which this quantity is negative, that is the controller is stable and makes the closed-loop system stable. In case it is unable to find such a controller, HIFOO terminates unsuccessfully.

In the performance optimization phase, HIFOO searches for a local minimizer of

$$f(K) = \begin{cases} \infty & \text{if } \max(\alpha(A_{CL}, \varepsilon\alpha(A_K))) \geq 0 \\ \max(\|T_{zw}\|_\infty, \varepsilon\|K\|_\infty) & \text{otherwise} \end{cases} \tag{7.4}$$

where

$$\|K\|_\infty = \sup_{Rs=0} \left\| C_K(sI - A_K)^{-1}B_K + D_K \right\|_2 \tag{7.5}$$

The introduction of ε is motivated by the fact that the main design objective is to attain closed-loop system stability and to minimize $\|T_{zw}\|_\infty$, by demonstrating that ε should be relatively small; the term $\varepsilon\|K\|_\infty$, however, prevents the controller H_∞ norm from becoming too large, in which case the stability constraint by itself would not exist. Given that it is preceded by the stability phase, the performance optimization phase is initialized with a finite value of $f(K)$. Consequently, when it reaches a value of K for which $f(K) = \infty$, that value is rejected, since an objective reduction is sought at each iteration [9, 20].

8 Results Using Controller HIFOO

As mentioned before, the HIFOO controller is implemented in Matlab by way of appropriate routines. It is called in the following manner:

Kfoo = hifoo(plant, 2)

where plant is the system description in the form of Eqs. (7.1), and $n = 2$ is controller order.

The resulting controller is described in state space in similar manner as H_∞, i.e.

$$\begin{aligned} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\ u(t) &= C_K x_K(t) + D_K y(t) \end{aligned} \tag{8.1}$$

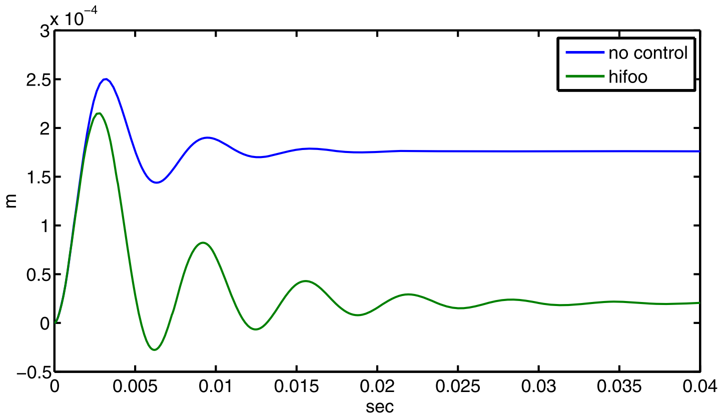


Fig. 16 Beam free end displacement, with and without HIFOO control, for the first mechanical input

The controller state space equation is given by relation (7.5), where controller matrices are equal to

$$A_K = \begin{bmatrix} 728.1 & -5034 \\ 207.5 & -1408 \end{bmatrix}$$
$$B_K = \begin{bmatrix} 212.8 & 811.6 & 1716 & 2810 \\ -164.9 & -637.2 & -1348 & -2207 \end{bmatrix}$$
$$C_K = \begin{bmatrix} 1557 & -916.7 \\ 1013 & -592.3 \\ 517 & -297.9 \\ 144.3 & -82.59 \end{bmatrix} \tag{8.2}$$
$$D_K = \begin{bmatrix} 36.1 & 136.6 & 287.1 & 468.3 \\ 23.5 & 87.69 & 186.5 & 303 \\ 12.12 & 44.12 & 93.39 & 154.3 \\ 4.204 & 12.53 & 26.92 & 43.51 \end{bmatrix}$$

For the purpose of comparison of HIFOO controller performance to that of H_∞, the beam free end response is examined, for the first and third mechanical input.

For the first mechanical input, equilibrium recovery time for the free end is 0.03 sec, as shown in Fig. 16, and maximum upward displacement is 2×10^{-4}, while steady state error is 10^{-7}. The maximum produced voltage for the control of the end node is 30 V, as shown in Fig. 17. Using H_∞ the free end restores equilibrium within 0.02 sec, as shown in Fig. 10, and its maximum upward displacement is 0.3×10^{-4}, while steady state error is 10^{-9}. The maximum produced voltage is 35 V, as shown in Fig. 12. Therefore, the HIFOO controller exhibits slightly inferior results with respect to all criteria, higher steady state error, longer equilibrium recovery time and larger upward displacement; however, it has lower energy demand and lower order.

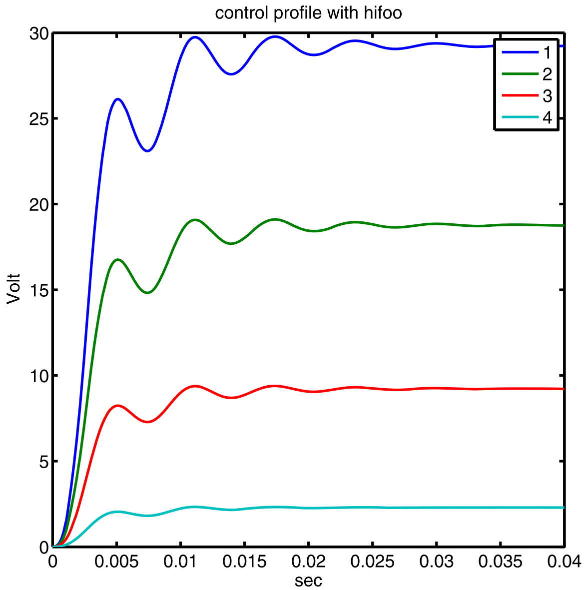


Fig. 17 Stress at beam nodal points, using HIFOO, for the first mechanical input

It can be observed that by using the HIFOO controller, a reduction of system controller order is achieved, while beam position is controlled with node displacements of order of 10^{-7}. The H_∞ criterion performance is thus retained while a lower order controller is used.

The steady state error is 3×10^{-5}, while using the H_∞ controller it is 0.5×10^{-5} (Fig. 16). The maximum produced voltage for the HIFOO controller is 30 V; the respective value is 35 V for the H_∞ controller [19]. In other words, beam adjustment to its equilibrium position is achieved with a lower order controller that requires lower voltage.

9 Conclusions

A finite element based modelling technique for the determination of the smart system was presented. Firstly we examine the H_∞ criterion. The advantage of the H_∞ criterion is in its ability to take into account in the computations the worst result of uncertain disturbances or noise in the system. It is possible to synthesize an H_∞ controller which will be robust with respect to a predefined number of uncertainties in the model. The results are very good: the oscillations were suppressed, with the piezoelectric components' voltages within their endurance limits.

Using nonsmooth and nonconvex optimization we reduce the order of the H_∞ controller. The good performance of the controller was made even for a lower order of the system. We reduce vibration with small recovery time and the piezoelectric patches in their endurance limits.

References

1. Balamurugan V, Narayanan S (2002) Finite element formulation and active vibration control study on beams using smart constrained layer damping (SCLD) treatment. J Sound Vib 249(2):227–250
2. Foutsitzi G, Marinova D, Hadjigeorgiou E, Stavroulakis G (2002) Finite element modelling of optimally controlled smart beams. In: 28th summer school: Applications of mathematics in engineering and economics, Sozopol
3. Foutsitzi G, Marinova D, Hadjigeorgiou E, Stavroulakis G (2003) Robust H2 vibration control of beams with piezoelectric sensors and actuators. In: Proceedings of physics and control conference (PhyCon03), St Petersburg, 20–22 August, vol I, pp 158–163
4. Miara B, Stavroulakis G, Valente V (2007) Topics on mathematics for smart systems. In: Proceedings of the European conference, Rome, 26–28 October 2006. World Scientific, Singapore
5. Moutsopoulou A, Pouliezos A, Stavroulakis GE (2008) Modelling with uncertainty and robust control of smart beams. In: Topping BHV, Papadrakakis M (eds) Proceedings of the ninth international conference on computational structures technology. Civil Comp Press, Stirling, Paper 35
6. Burke J, Overton ML (2001) Variational analysis of non-Lipschitz spectral functions. Math Program 90:317–352
7. Burke J, Lewis A, Overton ML (2005) A robust gradient sampling algorithm for nonsmooth nonconvex optimization. SIAM J Optim 15:751–779
8. Burke J, Henrion D, Lewis A, Overton ML (2006) Stabilization via nonsmooth, nonconvex optimization. IEEE Trans Autom Control 51:1760–1769
9. Burke J, Henrion D, Lewis A, Overton ML (2006) HIFOO a Matlab package for fixed-order controller design and H_∞ optimization. In: IFAC symposium on robust control design, Toulouse. http://www.cs.nyu.edu/overton/papers/pdffiles/hifoo-rocond.pdf
10. Hinrichsen D, Pritchard AJ (2005) Mathematical systems theory I: modelling, state space analysis, stability and robustness. Texts in applied mathematics, vol 48. Springer, New York
11. Packad A, Doyle J, Balas G (1993) Linear, multivariable robust control with a m perspective. J Dyn Syst Meas Control 115(2b):310–319 (50th anniversary issue)
12. Pouliezos A (2006) Mimo control systems, class notes. http://pouliezos.dpem.tuc.gr
13. Stavroulakis GE, Foutsitzi G, Hadjigeorgiou E, Marinova D, Baniotopoulos CC (2005) Design and robust optimal control of smart beams with application on vibrations suppression. Adv Eng Softw 36(11–12):806–813
14. Friedman J, Kosmatka K (1993) An improved two node Timoshenko beam finite element. Comput Struct 47:473–481
15. Tiersten HF (1969) Linear piezoelectric plate vibrations. Plenum, New York
16. Zhang N, Kirpitchenko I (2002) Modelling dynamics of a continuous structure with a piezoelectric sensor/actuator for passive structural control. J Sound Vib 249:251–261
17. Sisemore C, Smaili A, Houghton R (1999) Passive damping of flexible mechanism system: experimental and finite element investigation. In: The 10th world congress of the theory of machines and mechanisms, Oulu, vol 5, pp 2140–2145
18. Yang SM, Lee YJ (1993) Optimization of non collocated sensor, actuator location and feedback gain and control systems. Smart Mater Struct 8:96–102
19. Zhang X, Shao C, Li S, Xu D (2001) Robust H_∞ vibration control for flexible linkage mechanism systems with piezoelectric sensors and actuators. J Sound Vib 243(1):145–155
20. Millston M (2006) HIFOO 1.5: structured control of linear systems with a non trivial feedthrough. MS thesis, Courant Institute of Mathematical Sciences, New York University