ing argument to the pair $(A_l+K_lC_l, C_l)$ where $K_l=\begin{bmatrix} K \\ 0 \end{bmatrix}$ and the assertion of the theorem for observability follows. The application of the preceding argument to the unstable eigenvalues of A_l completes the proof.

Consider the case when the inputs are not directly observed in the output (i.e., $D=0$). Since $\rho(C[I-(A+KC)]^{-1}B) \leq \min[\rho(C), \rho(B)] \leq \min[n,m,p]$, Theorem 3.1 shows that at least p sensors are needed in order to detect simultaneous jumps in p different actuators when $p<n$ and $D=0$. Although there is practically no limit to the number of single actuator jumps, we can hypothesize as long as the necessary and sufficient rank condition is satisfied; the preceding theorem establishes a limit on the number of simultaneous actuator jumps that can be detected for a given number of sensors. When the outputs contain direct measurements of the inputs (i.e., $D \neq 0$), we have $\rho(C[I-(A+KC)]^{-1}[B+KD]+D) \leq \min[\rho(C), \rho(B+KD)]+\rho(D)] \leq \min[m,n]+\rho(D)$ so that at least $p-\rho(D)$ sensors are needed to detect p simultaneous jumps in inputs. That is, the necessary number of sensors to detect jumps in p inputs decreases by the number of linearly independent direct input measurements.

We also note that, in general, the matrix B in (2.1) will not be the system input matrix. If the original system input matrix is the $n \times l$ matrix B_0 and the hypothesized jump directions are $[f_1, f_2, \cdots, f_p]$ where $p<l$ and $f_i \in R^l$, then B will have the form $B_0 f_i$ for a single jump and the rank condition will be $\rho(C[I-(A+KC)]^{-1}[B_0 f_i+KD]+D)=1$. On the other hand, for simultaneous jumps in p actuators B will have the form $B=B_0F$ where $F=[f_1 f_2 \cdots f_p]$ and the rank condition will be given by $\rho(C[I-(A+KC)]^{-1}[B_0F+KD]+D)=p$.

IV. Detectability of Jumps in Outputs

Detectability of a jump in output in (2.1)–(2.3) will be dictated by the observability (detectability) of the pair of (A_0, C_0)

$$A_0=\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \qquad C_0=[C \quad H] \tag{4.1}$$

where I is the identity matrix of order q.

Theorem 4.1: The pair (A_0, C_0) defined by (3.1) is observable (detectable) iff (A,C) is observable (detectable) and $\rho(C[I-(A+KC)]^{-1}KH+H)=q$, where K is any matrix such that $1 \notin s(A+KC)$.

The proof follows from Theorem 3.1. If the system does not have neutrally stable eigenvalues, the condition for detectability of a jump in output reduces to the requirement that the original system be detectable and $\rho(H)=q$. Clearly, the detectability of an output jump puts less stringent requirements on the system than the detectability of an input jump.

V. Simultaneous Jumps in Inputs and Outputs

When simultaneous jumps in inputs and outputs are hypothesized, the observability (detectability) of the pair (A_{l0}, C_{l0})

$$A_{l0}=\begin{bmatrix} A & B & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \qquad C_{l0}=[C \quad D \quad H] \tag{5.1}$$

will be required for detection.

Theorem 5.1: The pair (A_{l0}, C_{l0}) as defined by (5.1) is observable (detectable) iff (A,C) is observable (detectable) and $\rho(C[I-(A+KC)]^{-1}[B+KD]+D|C[I-(A+KC)]^{-1}KH+H)=p+q$.

The proof is similar to that of Theorem 3.1 and is omitted.

VI. Detectability of Jumps in States

Another important jump phenomenon is an abrupt change in states. The detection of these types of jumps is especially important since they usually model unknown physical properties of a given system. The detectability of a state jump will be dictated by the observability of the pair

$$A_s=\begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} \qquad C_s=[C \quad 0]. \tag{6.1}$$

Theorem 6.1: The pair (A_s, C_s) defined by (6.1) will be observable iff (A,C) is observable and $\rho[C(A+KC)^{-1}E]=r$, where K is any matrix such that $A+KC$ is nonsingular.

The proof is similar to that of Theorem 3.1. Since E will represent a jump in states constrained to lie in an r-dimensional subspace [15] and since $\rho[C(A+KC)^{-1}E] \leq \min(m,n,r)$, at least r sensors are needed in order to detect a jump in states constrained to lie in an r-dimensional subspace.

VII. Conclusions

Necessary and sufficient conditions for detectability of jumps in inputs, outputs, and states in linear systems have been given. Results indicate the minimum number of sensors necessary for detection of different hypothesized jumps.

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An Iterative Method for Calculating the Sample Serial Correlation Coefficient

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***Abstract*—In this paper a fast on-line procedure for the calculation of the sample serial correlation for serially received data is presented.**

I. Introduction

In this brief paper an iterative procedure for the calculation of the serial correlation coefficient of lag k is proposed. This function is required in some applications of control theory, especially in stochastic

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systems where the Kalman filter is employed. The function may be used to check the optimality of the Kalman filter [1], [2].

It has been shown [2] that if possible errors in the description of the system model arise only from incorrect noise covariances, then the filter is optimal if and only if the measurement residuals are independent in time. A test can then be devised for checking this condition by looking at the history of the correlations.

As outlined in [3], this procedure is rather cumbersome computationally, particularly for on-line applications.

In the sequel, an iterative method is presented which is suitable for on-line batch processing of residuals.

II. Problem Statement

The sample serial correlations of lag k are calculated from

$$\hat{C}_k = \frac{1}{N}\sum_{i=k}^{N}(\beta_i-\hat{\beta})(\beta_{i-k}-\hat{\beta})^T \tag{1}$$

where N is the number of vector Kalman filter residuals β_i considered at each time instant and

$$\hat{\beta} = \frac{1}{N}\sum_{i=1}^{N}\beta_i$$

denotes the sample mean. (As mentioned in [4], division by N instead of $N-k$ in (1) results in estimates which are positive definite, asymptotically unbiased, and have smaller MSE.)

As mentioned before, an optimality test for the Kalman filter can be devised which utilizes the $\hat{C}_k$'s, $k=1,\cdots,l$. The test consists of looking at the diagonal elements of $\hat{C}_k$, $k=1,\cdots,l$ and counting the the number of times they lie outside the band $\pm 1.96/\sqrt{N}\,[\hat{C}_0]_{ii}$, for the 95 percent confidence limit.

If the number is greater than 95 percent of the total, the null hypothesis of normality of the serial correlations is rejected and a fault declared.

III. Computational Procedure

Suppose that at time instant t, when new information arrives, a batch of N residuals is considered, so that if

$$\beta_t,\beta_{t+1},\cdots,\beta_{t+N-1}$$

are considered at time t

$$\beta_{t+1},\beta_{t+2},\cdots,\beta_{t+N}$$

are considered at time $t+1$.

Let the serial correlations and means calculated at time t be denoted by

$$\hat{C}_k^t,\hat{\beta}^t$$

and their elements by

$$[\hat{C}_k^t]_{ii},\hat{\beta}_i^t.$$

In the proposed method $[\hat{C}_k^{t+1}]_{ii}$ is calculated from $[\hat{C}_k^t]_{ii}$. This is done by first calculating $[\hat{C}_0^{t+1}]_{ii}$ from $[\hat{C}_0^t]_{ii}$ and then $[\hat{C}_k^{t+1}]_{ii}$ from $[\hat{C}_{k-1}^{t+1}]_{ii}$, $k=1,\cdots,l$.

The iteration on k is considered first. From (1)

$$[\hat{C}_{k-1}^t]_{aa} = \frac{1}{N}\sum_{i=t-N+k}^{t}(\beta_i-\hat{\beta}^t)_a(\beta_{i-k+1}-\hat{\beta}^t)_a$$

$$= \frac{1}{N}\sum_{i=t-N+k}^{t}\left(\beta_i\beta_{i-k+1}-\beta_i\hat{\beta}^t-\hat{\beta}^t\beta_{i-k+1}-(\hat{\beta}^t)^2\right)_{aa}. \tag{2}$$

Also

$$[\hat{C}_k^t]_{aa} = \frac{1}{N}\sum_{i=t-N+k+1}^{t}\left(\beta_i\beta_{i-k}-\beta_i\hat{\beta}^t-\hat{\beta}^t\beta_{i-k}-(\hat{\beta}^t)^2\right)_{aa}. \tag{3}$$

Comparing terms in (2) and (3) and dropping suffixes gives

$$\hat{\beta}^t\sum_{i=t-N+k+1}^{t}\beta_i = \left(\hat{\beta}^t\sum_{i=t-N+k}^{t}\beta_i\right)-\beta_{t-N+k} \triangleq q_k^t$$

$$\hat{\beta}^t\sum_{i=t-N+k+1}^{t}\beta_{i-k} = \left(\hat{\beta}^t\sum_{i=t-N+k}^{t}\beta_{i-k}\right)-\beta_{t-k+1} \triangleq p_k^t$$

and

$$\sum_{i=t-N+k+1}^{t}(\hat{\beta}^t)^2 = \left(\sum_{i=t-N+k}^{t}(\hat{\beta}^t)^2\right)-(\hat{\beta}^t)^2 = (N-k)(\hat{\beta}^t)^2.$$

Therefore

$$[\hat{C}_k^t]_{aa} = \frac{1}{N}\left[(N-k)(\hat{\beta}^t)^2-(q_k^t+p_k^t)+\sum_{i=t-N+k+1}^{t}\beta_i\beta_{i-k}\right] \tag{4}$$

where

$$q_k^t = q_{k-1}^t-\beta_{t-N+k}$$

$$p_k^t = p_{k-1}^t-\beta_{t-k+1}$$

initialized by

$$q_0^t = p_0^t = \sum_{i=t-N+1}^{t}\beta_i = N\hat{\beta}^t. \tag{5}$$

So once q_0^t, p_0^t, and $\hat{\beta}^t$ are evaluated, $\hat{C}_k^t$ can be calculated for each $k=1,\cdots,1$. From (5) follows that it is sufficient to calculate $\hat{\beta}^t$ from $\hat{\beta}^{t-1}$. This can be easily done by considering

$$\hat{\beta}^t = \sum_{i=t-N+1}^{t}\beta_i = \hat{\beta}^{t-1}+\frac{1}{N}(\beta_t-\beta_{t-N}). \tag{6}$$

It remains to calculate $\hat{C}_0^t$ from $\hat{C}_0^{t-1}$. From definition

$$\hat{C}_0^t = \frac{1}{N}\sum_{i=t-N+1}^{t}(\beta_i-\hat{\beta}^t)^2 \tag{7}$$

$$\hat{C}_0^{t-1} = \frac{1}{N}\sum_{i=t-N}^{t}(\beta_i-\hat{\beta}^{t-1})^2. \tag{8}$$

From (6)

$$\hat{C}_0^t = \frac{1}{N}\sum_{i=t-N+1}^{t}\left(\beta_i^2+(\hat{\beta}^t)^2-2\beta_i\hat{\beta}^t\right)$$

$$= \frac{1}{N}\left[\sum_{i=t-N+1}^{t}\beta_i^2-2\hat{\beta}^t\sum_{i=t-N+1}^{t}\beta_i+\sum_{i=t-N+1}^{t}(\hat{\beta}^t)^2\right].$$

Comparing with (8)

$$\frac{1}{N}\sum_{i=t-N+1}^{t}\beta_i^2 = \left(\frac{1}{N}\sum_{i=t-N}^{t}\beta_i^2\right)+\frac{1}{N}(\beta_t^2-\beta_{t-N}^2).$$

Also

$$2\hat{\beta}^t\sum_{i=t-N+1}^{t}\beta_i = 2\hat{\beta}^tN\hat{\beta}^t = 2N(\hat{\beta}^t)^2$$

and

$$\sum_{i=t-N+1}^{t}(\hat{\beta}^t)^2 = N(\hat{\beta}^t)^2.$$

Now

$$(\hat{\beta}^t)^2 = \left(\hat{\beta}^{t-1}+\frac{1}{N}(\beta_t-\beta_{t-N})\right)^2$$

$$= (\hat{\beta}^{t-1})^2+\left(\frac{1}{N}(\beta_t-\beta_{t-N})\right)^2+\frac{2\hat{\beta}^{t-1}}{N}(\beta_t-\beta_{t-N}).$$

Hence

$$\hat{C}_0^t = \hat{C}_0^{t-1} + \frac{1}{N}(\beta_t^2 - \beta_{t-N}^2) - (\beta_t - \beta_{t-N})^2 - 2\hat{\beta}^{t-1}(\beta_t - \beta_{t-N}).$$

Let $a^t = 1/N(\beta_t - \beta_{t-N})$, then

$$\hat{C}_0^t = \hat{C}_0^{t-1} + \frac{1}{N}(\beta_t^2 - \beta_{t-N}^2) - a^t(a^t + 2\hat{\beta}^{t-1}).$$

So once the first batch of N residuals has been processed and $\hat{C}_0^N$ has been calculated, an iteration may proceed for subsequent batches.

The algorithm is suitable for implementation on a digital computer, and results in a faster execution time with reduced storage requirements by comparison with existing methods.

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A Projected Stochastic Approximation Method for Adaptive Filters and Identifiers

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***Abstract*—Generally, when stochastic approximation is used to identify the coefficients of a linear system or for an adaptive filter or equalizer, the iterate X_n is projected back onto some finite set $G = \{x : |x_i| < B, \text{ all } i\}$, if it ever leaves it. The convergence of such truncated sequences have been discussed informally. Here it is shown, under very broad conditions on the noises, that $\{X_n\}$ converges with probability 1 to the closest point in G to the optimum value of X_n. Also, under even weaker conditions, the case of constant coefficient sequence is treated and a weak convergence result obtained. The set G is used for simplicity. It can be seen that the result holds true in more general cases, but the box is used since it is the only commonly used constraint set for this problem.**

I. Introduction

Reference [1] deals with a great variety of stochastic approximation procedures, for constrained and unconstrained systems and for convergence with probability 1 (w.p.1) and weak convergence,[1] all for systems with correlated inputs. The techniques of [1] are readily usable for many problems that are not explicitly treated there. This will be illustrated here for one particular class of constrained problems which is of great current interest and which arises in identification and in adaptive control theory. In fact, it is just such constrained problems to which more attention should be given, owing to their prevalence. The proofs are contained in various parts of [1] and, here, after the problem is defined, it is shown how to put the bits and pieces together. The problem and method are typical of a large class of adaptive systems which can be treated by similar methods, and is worth singling out.

In [1], [2], it is shown how (under certain conditions) an ordinary differential equation can be associated with a stochastic approximation algorithm, and that the asymptotic points of the SA (constrained or not) are the stability points or invariant sets of this equation (equation projected onto the constraint set in the constrained case). Such an idea is very useful but has not been explicitly proven for the class of algorithms dealt with here. This extension is done here. The methods are applicable to a broader class of problems. In our case, the limit point will obviously satisfy the necessary condition for a minimum in the "least square" identification problem—indeed, there is only one point satisfying this condition. Previous works on constrained SA have not dealt explicitly with algorithms where the noise appears in the form that it does here. A purpose of this paper is to show how to make the necessary adjustments in the proofs so that such algorithms are covered.

The problem will be set up in such a way that it fits both a standard identification problem and a standard problem in adaptive equalizers. Let $\{d_n\}$ denote a scalar valued desired output sequence, perhaps a training or reference signal, or output of the system to be identified. The problem can readily be set up so that all quantities $\{d_n, u_n, X_{ni}, \rho_n\}$ are complex valued, but in the interest of simplicity, we suppose that they are real valued. Let $\{u_n\}$ denote an input sequence set $\psi_n \equiv (u_n, \cdots, u_{n-r+1})'$ and let $\{\rho_n\}$ be a noise sequence, independent of $\{u_n\}$. The observed adaptive system output at time n is defined by $\sum_{i=0}^{r-1} X_{ni} u_{n-i} = y_n$, and the "perturbed" observed reference at time n is $d_n + \rho_n$. The idea in [2]–[4] and in many other papers is to adjust the system parameter $X_n = (X_{n0}, \cdots, X_{n,r-1})'$ so that the output $\{y_n\}$ "best matches" the $\{d_n\}$ in a mean-square sense. A common recursive adaptive algorithm for doing this is

$$\begin{aligned} X_{n+1} &= X_n - a_n\psi_n\epsilon_n, \qquad \epsilon_n = (y_n - d_n - \rho_n) \\ &= X_n - a_n\psi_n(\psi_n' X_n - d_n - \rho_n), \qquad a_n \to 0, \qquad \sum_n a_n = \infty, \qquad a_n > 0. \end{aligned} \tag{1.1}$$

Algorithm (1.1) has been the focus of an enormous amount of effort. In practice, there is usually given a bound B such that if some $|X_{ni}| > B$, then X_{ni} is immediately reset to the closest value $+B$ or $-B$. This projected version has received little attention. Ljung [2] discusses it but deals with it only when the optimum value of X_n is *strictly inside* the box $G = \{x : |x_i| < B_i\}$. The methods of [1] can readily handle such problems whether or not the unconstrained optimum is in G. Assumptions are stated in Section II. These are of the type used in [1] and are quite unrestrictive. In Section III it is shown that $\{X_n\}$ converges w.p.1 (under assumptions in Section II) to the point in G which is *closest* to the optimum value. Incidentally, if the optimum is strictly interior to G, then the rate of convergence results in [5] hold. Section IV deals with a formulation where $a_n \equiv \beta > 0$, *a constant*, and discusses some limit results of a "weak convergence" nature, also using techniques from [1].

In many of the proofs in [1], it is assumed that the iterate sequence $\{X_n\}$ is bounded in some sense. Owing to the possible use of the projection algorithm (as in this paper), this boundedness assumption is hardly a restriction. This is one of the secondary points of this paper.

II. Assumptions for the Probability One Convergence Case

Define $m(t) = \max\{n; t_n \le t\}$, $t > 0$, where

$$t_n = \sum_{i=0}^{n-1} a_i, \; m(t) = 0, \text{ for } \quad t < 0.$$

A1) *There is a positive definite symmetric matrix R such that for each $\epsilon > 0$ and some $T < \infty$*

$$\lim_{n\to\infty} P\left\{ \sup_{j>n} \max_{t<T} \left| \sum_{i=m(jT)}^{m(jT+t)-1} a_i(\psi_i\psi_i' - R) \right| > \epsilon \right\} = 0. \tag{2.1}$$

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[1] Weak convergence is a substantial extension of convergence in distribution [1]. Roughly, a sequence $\{P_n\}$ of measures on the space of continuous functions $C[0,\infty)$ converges weakly to a measure P on $C[0,\infty)$, if for each real valued function F which is bounded and continuous on $C[0,\infty)$, $\int F(f)dP_n(f) \to \int F(f)dP(f)$.

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