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# FATIGUE LIFE PREDICTION USING A NEW MOVING WINDOW REGRESSION METHOD

G. S. STAVRAKAKIS AND A. POULIEZOS

*Department of Electronic Engineering and Department of Production and Management Engineering, Technical University of Crete, Crete, Greece*

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A simple method to assess the safety of metallic structures in the presence of crack-like defects is considered. Linear regression techniques are used to fit fatigue-crack growth (FCG) laws of exponential type with respect to the number of cycles, in FCG experimental results. A moving window regression method is used to accurately predict whether or not the critical crack length region is reached. The probability of fast failure at any time (number of cycles) is formulated in terms of a characteristic defect size exceeding a critical value. The concept of conditional probability of failure is also employed here in order to evaluate the failure rate used in reliability practice. As an illustrative example, an analytical closed-form solution for the failure probability of structures is presented and evaluated using experimental FCG data.

## 1. INTRODUCTION

During the past twenty years extensive efforts have been devoted to developing techniques that permit the prediction of fatigue life of metallic structures [1-6, 9, 10]. As the knowledge related to fatigue of structures and materials expanded, it became clear that in certain cases fatigue could be treated from a propagation point of view. This knowledge has led to increased life of structures subjected to dynamic loads [7, 8, 10].

Any structural reliability assessment of metallic structures can be thought of as a global procedure relying on a number of steps or single procedures to be concatenated: non-destructive testing (NDT), material characterisation, analysis of loads, stress analysis, fracture mechanics analysis, fatigue crack growth analysis, failure analysis. Thus, the fatigue crack growth (FCG) and failure analysis presented here, should be considered as part of this overall assessment.

The fatigue process in metallic structures can be divided into stages of cycles to initiation and cycles to failure [1, 2, 12, 13]. The microscopic mechanism of each of the above will not be dealt with at this time. As the crack grows, it goes through a transition from a flat, plane strain fracture mode, to a slant, plane stress, fracture mode. The evaluation of fatigue-crack propagation behaviour is performed to compare materials and to experimentally (empirically) and analytically develop fatigue-crack propagation prediction capability. Numerous fatigue-crack propagation "laws" have been proposed. A large number of these are summarised in [9]. All analytical approaches to the problem have attempted to relate the growth rate ($da/dN$) to crack length, plastic enclave size, material constants, stress, and specimen dimensions.

The purpose in building models of FCG is to be able to accurately predict future macroscopic behaviour for purposes of engineering design and reliability management. Thus, we seek a state vector in macroscopic observables that defines material damage in FCG and a set of equations that describe the evolution of the state vector such that, given the value of the state vector at time $t_0 = 0$, we can predict its value for any $t \geq t_0$ independent of past history.



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If a model depends on details of past history we cannot predict the future evolution knowing only the present, and we consider this to be a weakness in any model [2, 3]. The set of equations must depend on the parameters that determine the conditions of loading and the pertinent features of material behaviour plus environment [1-5, 10].

A model of a physical phenomenon is deemed acceptable if it is consistent with the known data and predicts correctly the evolution of the phenomenon even under conditions different from those pertaining to the current observed data [2, 4, 12, 13].

The FCG process is an irreversible non-decreasing dynamic process with non-reproducible sample functions. Three sources of variability enter crack growth test data: one is the difference in material behaviour among identically prepared specimens or components, the second is difference in environment among tests at the same load condition and with the same material and finally differences occur due to equipment and personnel differences.

As is well known, models of any physical phenomenon whose evolution in time is governed by probabilistic laws, are stochastic processes. One natural way to construct such models is through dynamic equations with random initial conditions, random parameters or randomly time-varying coefficients.

The procedure that has been followed first in many investigations is based upon a randomisation of the Paris–Erdogan one-dimensional state equation [1, 7, 12]. The randomisation of the Paris–Erdogan equation assumes that the parameters are time independent and uncorrelated random variables, which is not really true [12, 13]. A model that contains a two-dimensional state vector is presented in [4]. We note that by increasing the dimension of the state vector from one to two, history dependence may be eliminated. Naturally, data would have to support this position.

In [5] the FCG analysis is carried out via diffusion process methods. The problem there is that, in general, the statistical properties of the solution process, i.e. the crack length, cannot be obtained unless some kind of approximations are made to the data. However, the fit to the theoretical distributions appear to exhibit significant deviations. In [6] FCG modelling is based upon a more detailed consideration of underlying microscopic level mechanisms. The evolution of microcracks is assumed to be governed by the scalar Ito equation. The associated distribution for the number of cycles to reach a given crack length is obtained as rather complicated functions of exponentials. In [10] a sophisticated stochastic model is presented describing the fatigue crack growth phenomenon under random overloads. The predicted lifetime is strongly dependent on the prescribed reliability level there: the more stringent the reliability requirement, the less beneficial the lifetime predictions.

Recent studies for FCG modelling employ Markoff chain models that generate probability distributions of damage accumulation directly and do not start from differential equations for sample functions of crack growth. The most intensively studied models of this type are B-models that employ Markoff chains [2, 3]. Crack length is the single observable employed. Parameter estimation has proved straightforward [2]. However, the B-model approach may be too cumbersome to yield insight into crack growth behaviour [2]. The tabulated values of the correlation coefficients for the random time required for the crack length to increase from level $(j-1)$ to level $(j)$ reveal that the FCG process is history dependent with positive correlation [2]. This means that the derivative $\mathrm{d}a/\mathrm{d}N$ is a function not only of $a$, but of $N$ also, a fact that is usually overlooked.

In the present paper FCG laws of crack length as an exponential function of the number of accumulated cycles are considered. These FCG laws date back to 1956 [11], but they are adopted here because they are simple in their application and provide good approximations to the actual experimental data. The derivative $\mathrm{d}a/\mathrm{d}N$ is a function of $N$ in these laws, thus the FCG prediction is history dependent. The simplicity of these FCG laws permits the application of linear regression techniques and thus the fast FCG phenomenon can be assessed using on-line inspection measurements under any loading (including random overloads) and temperature conditions. This is characteristic of simple FCG models only. Sophisticated models, as those in [2-6, 9, 10], cannot be used for such purposes. The lack of precision of simple models in assessing the FCG phenomenon is not a disadvantage, because their parameters are updated continuously using a recursive moving window technique. Thus, these adaptively identified models, represent precisely the physical FCG phenomena at any time instant independently of the loading, environmental and material conditions. Moreover, they are very convenient if the rational probabilistic approach to calculate the structure failure probability is used. A set of experimental data is used to validate the above considerations.

## 2. SIMPLE PROBABILISTIC MODELS OF FATIGUE CRACK GROWTH

There is a considerable experimental evidence that the crack length $a(N)$ as a function of the number of accumulated cycles can be modelled by the following three forms [11]:

$$a(N) = C_1 N^{m_1} \tag{1}$$

$$a(N) = C_2(\log_{10} N)^{m_2} \tag{2}$$

$$a(N) = C_3\, e^{m_3 N} \tag{3}$$

where $C_i$ and $m_i$, $i = 1, 2, 3$, are functions of applied load, material characteristics, geometrical configuration of the component and the initial quality of the product being tested.

When several identical components are tested under nominally identical environmental and operating conditions, the observed curves (realisations or sample functions), show a considerable scatter, as illustrated in Fig. 1 [13].

If each realisation is modeled by equations (1), (2) or (3), then the random function defining the stochastic phenomena of FCG can be expressed mathematically by equations (1), (2) and (3), where $a(N)$ exhibits stochastic behaviour due to one of the following

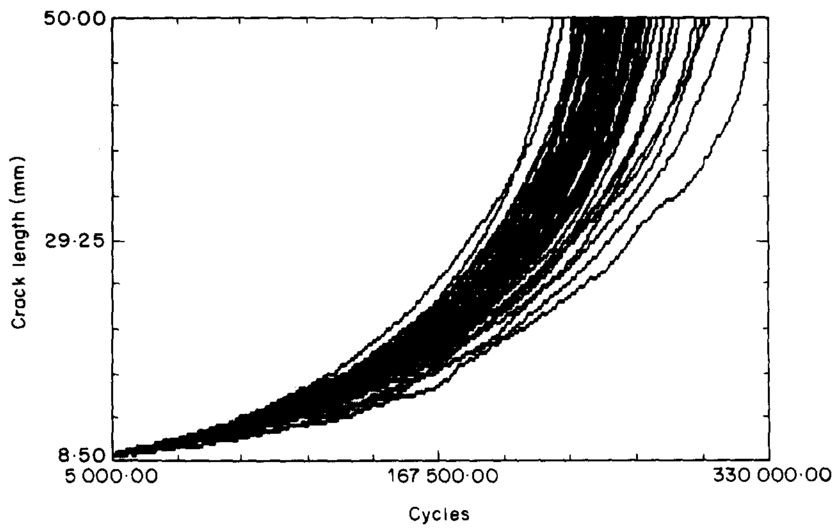


Figure 1. Virklers data, lin/lin.

reasons: (i) both $C_i$ and $m_i$ are random variables; (ii) at least one of them is random in nature.

The nature and range of variability of $a(N)$ for the experimental data of Fig. 1, can be seen in the frequency table of Fig. 2 at $N=2\times10^5$ cycles. A statistical investigation of the fatigue crack propagation process was conducted in [13]. The log-normal distribution provided the best fit for the cycle count data as evidenced by the low distribution ranking value and the very large number of times it was selected there as the best distribution.

Equations (1), (2), (3) can be rewritten as:

$$\ln a(N)=\ln C_1+m_1 \ln N \tag{1a}$$

$$\ln a(N)=\ln C_2+m_2 \ln [\log_{10} N] \tag{2a}$$

$$\ln a(N)=\ln C_3+m_3 N = C^*+m_3 N. \tag{3a}$$

The graphs of equations (1a), (2a) and (3a) are shown respectively in Figs 3, 4 and 5. The curves representing these equations should look like straight lines if the underlying

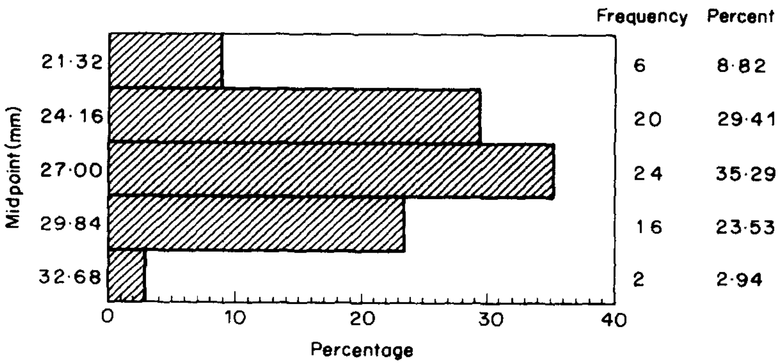


Figure 2. Frequency table at $N=2\times10^5$ cycles.

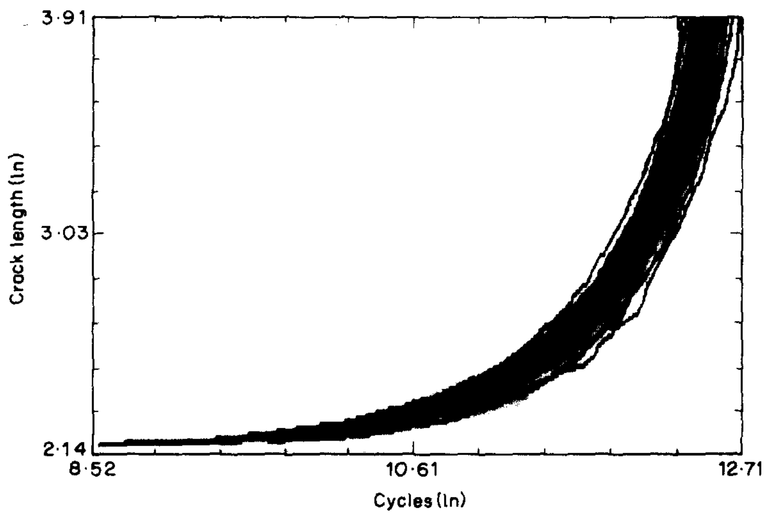


Figure 3. Virklers data, log/log equation (1a).

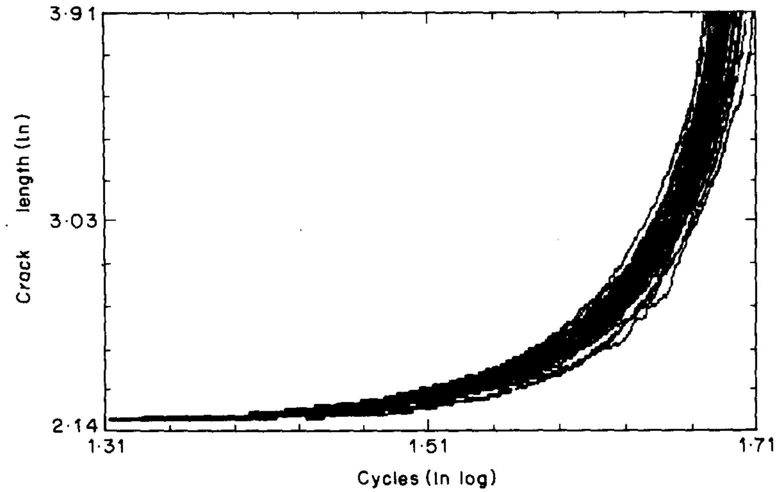


Figure 4. Virkler's data equation (2a).

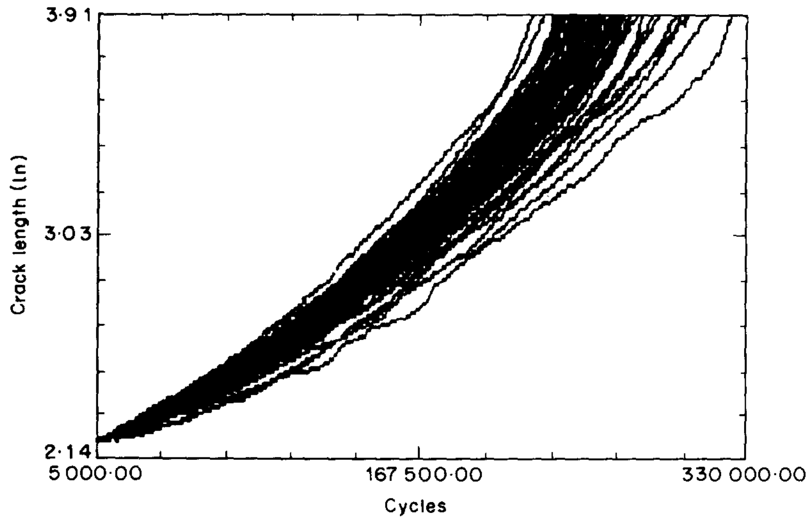


Figure 5. Virkler's data, equation (3a).

laws are correct. Simple inspection shows that equation (3a) provides the best approximation for the specific sets of available data.

## 3. THE FAILURE PROBABILITY PREDICTION MODEL

The basic problem in structural design is to ensure that an inadmissible failure state will not occur during the lifetime of the structure. It is thus required that the crack length, $a(N)$, does not exceed a critical value, $a_c$. Let the thermomechanical state of the structure be characterised, at any cycle number $N$, by a finite number of parameters governing applied load, material characteristics, geometrical configuration, temperature, etc., conditions. All the above thermomechanical state factors are lumped into the linear FCG equation.

The basic design problem is then to determine, at any number of cycles $N$, the probability of structural failure, i.e. the probability $U(N)$ that an inadmissible failure state will occur

$$U(N) = P[a(N) \geqslant a_c] = 1 - P[a(N) < a_c] = 1 - R(N) \tag{4}$$

where $R(N)$ is the structural reliability function.

Linear regression techniques can be easily applied to fit equations (1a), (2a) and (3a) in real FCG data and to assess the variability of the parameters $C_i$ and $m_i$, $i = 1, 2, 3$.

Equation (3a) will be used for the rest of the paper since it provides the best fit as explained earlier. Moreover the number of cycles appears explicitly with any logarithmic transformation. The parameters in equation (3a) can be updated after any inspection of the structure using the method described in Section 4.

Thus, if the random variable $a(N)$ has a log-normal distribution, its natural logarithm follows a normal distribution. It follows from equation (3a) that the quantity $C^* = \ln C_3$ will be normally distributed with mean $E[C^*]$ and variance $\mathrm{var}[C^*]$ if the parameter $m_3$ has zero variability or if it is also normally distributed.

One way to estimate the parameter $m_3$ as a deterministic value (i.e. with zero variability) is to use the following procedure

(1) Estimate the values of $C_i^*$, $m_{3,i}$, $i = 1, 2, \ldots, n$ for all $n$ original replicate ($\ln a$ vs. $N$) data.

(2) Find the mean value of $\bar{m}_3 = 1/n \sum_{i=1}^{n} m_{3,i}$ ($\bar{m}_3 > 0$).

(3) Re-estimate the values of $C_i^*$ for all original replicate ($\ln(a)$ vs. $N$) pairs of data considering the value of $m_3$ constant and equal to $\bar{m}_3$. A histogram for all these $C_i^*$ values can be constructed in order to assess the variability of applied load, geometrical configuration and material properties. The values of $E[C^*]$ and $\mathrm{var}[C^*]$ are then estimated from the histogram using the well known formulae from elementary statistics.

For engineering calculations both $\bar{m}_3$, considered deterministic and estimated by the above procedure, and $C^* = \ln C_3$ can be assumed to be statistically independent. Thus

$$R(N) = P[a(N) < a_c] = P[\ln a(N) < \ln a_c] = \phi\left(\frac{\ln a_c - E[\ln a(N)]}{\{\mathrm{var}[\ln a(N)]\}^{1/2}}\right)$$

where

$$\phi(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{u} \mathrm{e}^{-u^2/2}\, \mathrm{d}u \tag{5a}$$

is the Laplace function.

From equation (3a) we can obtain

$$E[\ln a(N) = E[C^*] + \bar{m}_3 N \tag{6}$$

$$\mathrm{var}(\ln a(N)] = \mathrm{var}[C^*] \tag{7}$$

Equation (7) implies that our assumptions on the statistical properties of $m_3$ and $C_3$ result in a variance for $a(N)$ which is independent of the number of cycles $N$. This means that for small $N$ the variance is overestimated, while for large $N$ it is underestimated, as illustrated in Fig. 6. This is not expected to have a significant effect on the reliability calculations, since it makes our estimates pessimistic in the critical crack length cycle range.

Substituting equations (6) and (7) in (5) and using the property of the Laplace function $\phi(-u) = 1 - \phi(u)$, the reliability function is given by:

$$R(N) = 1 - \phi\left(\frac{N - A}{B}\right), \qquad N > 0 \tag{8}$$

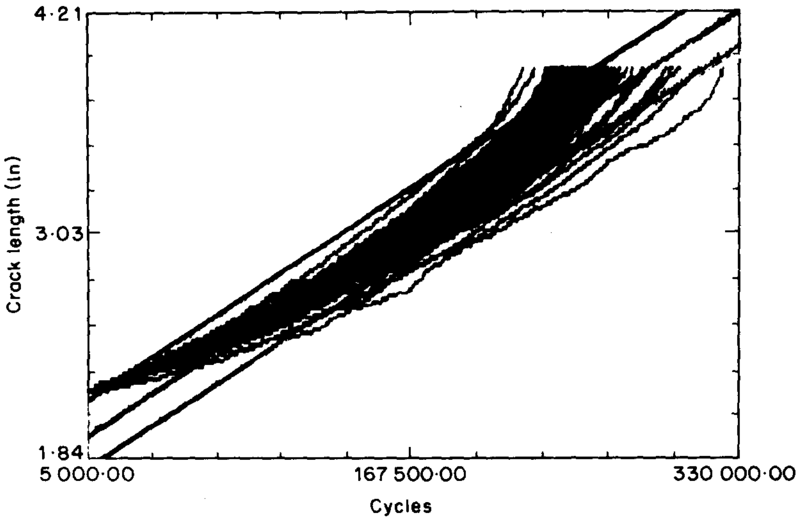


Figure 6. Mean graph with variance limits on Virkler's data, equation (3a).

where the parameters $A$, $B$ are:

$$A=\frac{\ln a_c - E[C^*]}{\bar{m}_3}, \qquad B^2=\frac{[\mathrm{var}\,(C^*)]}{\bar{m}_3^2}. \tag{9}$$

Parameter $A$ represents an estimation of the mean number of cycles in order to attain the critical crack length $a_c$.

Parameter $B$ determines the role of the quality of the product, i.e. variability of the properties of the material and of the loading and thermal conditions, or the measurement error introduced by the crack detection method, [12, 13].

From equations (4) and (8) it follows immediately that:

$$U(N)=\phi\left(\frac{N-A}{B}\right). \tag{10}$$

The corresponding probability density function (PDF) is

$$f(N)=-\frac{\mathrm{d}R(N)}{\mathrm{d}N}=\frac{1}{(2\pi)^{1/2}B}\exp\left[-\frac{1}{2}\left(\frac{N-A}{B}\right)^2\right]=\varphi\left(\frac{N-A}{B}\right). \tag{11}$$

Equation (11) is the PDF of the normal distribution with parameters $\mu=A$ and $\sigma=B$. The widespread applicability of the normal distribution in fatigue life studies is well known and it is verified by the above result [12–14].

The hazard rate function $h(N)$ is the conditional probability of failure at $N$ cycles given that failure has not occurred before. In other words, $h(N)$ is the instantaneous failure rate at $N$ cycles. From the classical reliability theory $h(N)$ for the normal distribution is given by:

$$h(N)=\frac{\varphi(u)}{1-\phi(u)}, \qquad u=\frac{N-A}{B}, \qquad N\geqslant 0. \tag{12}$$

The failure rate is an increasing function of time in the present case. This is the reason that the normal distribution is appropriate to wear-out and fatigue accumulation types of failure [13, 14].

Approximate expressions for the fast computation of the above reliability characterisations are given in Appendix A.

## 4. A MOVING WINDOW REGRESSION METHOD TO ASCERTAIN STRUCTURAL SAFETY

In this section a simple method is developed for detecting the number of cycles after which the damage propagates quickly. Thus, the structure becomes unsafe and appropriate action should be taken.

From Fig. 1 it can be seen that the curves may be thought of as consisting of two parts. One part corresponds to the slow propagation of the damage, while the other corresponds to the fast one. However it is not apparent where the dividing point is. By considering Fig. 4 one can see that the logarithmic transformation of the data makes this division clearer. Thus, by a simple visual inspection of the observed $\ln(a(N))$ vs. $\ln(\log_{10} N)$ graph it becomes possible to identify the dividing value of the number of cycles. However, equation (2a) is inadequate for accurate FCG prediction purposes because it cannot model satisfactorily the curves of Fig. 4. As mentioned earlier equation (3a) is the most suitable for this goal despite the fact that the curves of Fig. 5 are not exactly straight lines. However, straight lines modelled by equation (3a) can adequately represent large portions of them. In particular, if one considers moving windows of data of appropriate length, iterative regression techniques can be used to track the varying slopes of the approximating straight lines.

In this way an adaptive prediction method is introduced which is especially desirable in such cases, since the parameters of equation (3a) change with time (number of cycles), due to the continuous variation of the conditions related with the FCG condition (stress transients, random overloads, temperature, material properties, inspection technique variability etc.)

Equation (3a) is rewritten as follows

$$\ln(a(N)) = [1 N]\begin{bmatrix}\ln C_3 \\ m_3\end{bmatrix} = \mathbf{u}^T\boldsymbol{\theta}.$$

If additionally $\ln(a(N)) = y$, then for $n$ pairs of $(a(N), N)$ points, the well known linear regression formula gives,

$$\hat{\boldsymbol{\theta}} = [\mathbf{U}^T\mathbf{U}]^{-1}\mathbf{U}^T\mathbf{y}$$

where $\mathbf{U}$, $\mathbf{y}$ hold the information for the whole set of data. To denote explicitly the dependence of the estimated parameters on the number of cycles, equation (3a) may be written more accurately as,

$$\mathbf{y} = \mathbf{u}^T\boldsymbol{\theta}(N).$$

Iterative methods that update the estimate whenever new information is available can also be used. For accurate detection purposes, a moving window regression formula is more appropriate, since it is more sensitive to parameter changes during the variation of the thermomechanical conditions of the structure. As shown in Appendix B, a moving window estimate is given by the following recursive equations

$$\hat{\boldsymbol{\theta}}(k+1) = \hat{\boldsymbol{\theta}}(k) - \mathbf{P}(k+1)[\boldsymbol{\Gamma}(k+1)\hat{\boldsymbol{\theta}}(k) - \boldsymbol{\delta}(k+1)]$$

$$\mathbf{P}^{-1}(k+1) = \mathbf{P}^{-1}(k) + \boldsymbol{\Gamma}(k+1)$$

where

$$\Gamma(k+1) = \mathbf{u}(k+1)\mathbf{u}^T(k+1) - \mathbf{u}(k-n_W+1)\mathbf{u}^T(k-n_W+1)$$

$$\boldsymbol{\delta}(k+1) = \mathbf{u}(k+1)y(k+1) - \mathbf{u}(k-n_W+1)y(k-n_W+1)$$

and $n_W$ is the window length.

The proposed FCG prediction algorithm consists of the following steps:

*Step* 1: Compute $\hat{\boldsymbol{\theta}}(n_W)$ for the first $n_W$ pairs of ($\ln a$, $N$) data, using the one-shot regression formula.

*Step* 2: Process the pair of data coming from the next inspection using the moving window regression formulae.

*Step* 3: Estimate the one step ahead predicted value for $a(N)$ using equation (3a) and $\hat{\boldsymbol{\theta}}(N)$. The value of $N$ (number of cycles) used in this one step ahead predictor is the number of cycles for the next inspection according to the inspection-maintenance schedule of the structure.

*Step* 4: The predicted value of $\hat{a}(N)$ is checked against the predetermined critical crack length threshold $a_c$. If $\hat{a}(N) \geq a_c$ an emergency condition is declared appropriate action should be taken, otherwise go to step 2.

## 5. PRACTICAL RESULTS AND DISCUSSION

To answer the structural reliability investigation objectives, it is necessary to conduct an experimental replicate test under identical load and environmental conditions to satisfy the statistical requirements of the test program.

We considered the experimental data of Virkler *et al.* [13], which comprises 68 replications with constant load amplitude cycling loading. The data consists of the number of cycles required to reach 164 crack lengths, starting at 9 mm and terminating at 49·8 mm, for each replication. Centre crack aluminium (2024-T3) test specimens were employed. The original replicate ($a$ vs. $N$) data are shown in Fig. 1.

The FCG law of equation (3a) is fitted into this data, using the linear regression technique described in Section 4. Using the procedure described in Section 3 the "deterministic" value of the parameter $m_3$ is estimated to be $6{\cdot}89 \times 10^{-6}$ and the mean value and variance of the parameter $C^*$ are estimated as 1·94 and $7{\cdot}67 \times 10^{-3}$ respectively.

The failure probability of a cracked aluminium-2024-T3 structure constructed using the aluminium corresponding to the loading conditions of Virkler's experiment can be calculated using equations (9) and (10). The failure probability for the Virkler's experiment at $N = 2 \times 10^5$ cycles and for a critical crack length $a_c = 32{\cdot}68$ mm, is found, by applying equations (9), (10), to be $\hat{U}(200\,000) = 0{\cdot}0274$. Parameters $A$, $B$ of equation (9) were found to be 224 958·15 and 12 715·5 cycles respectively. From the propagated crack length histogram at $N = 2 \times 10^5$ cycles derived directly from the Virkler *et al.* experiment the same probability is evaluated as $U(200\,000) = 0{\cdot}0294$. This represents a discrepancy of 6·8%.

The usefulness of the moving window method is illustrated using one set of Virkler's data. Simulation runs for the one-step ahead predictor indicated that the optimum window length is $n_W = 4$. This produced a maximum absolute prediction error of 0·23 over the whole range of data, as shown in Fig. 7. If predictions of longer horizon are required, simulation runs could establish the corresponding optimum window length.

The application of the above procedures for the safety and failure probability assessment of a real structure is straightforward. Consider first an individual defect. Let the thermomechanical state of the structure, in the neighbourhood of the defect be characterised at any number of cycles $N$, by a finite number of physical parameters $q_i$, $i = 1, 2, \ldots, n$.

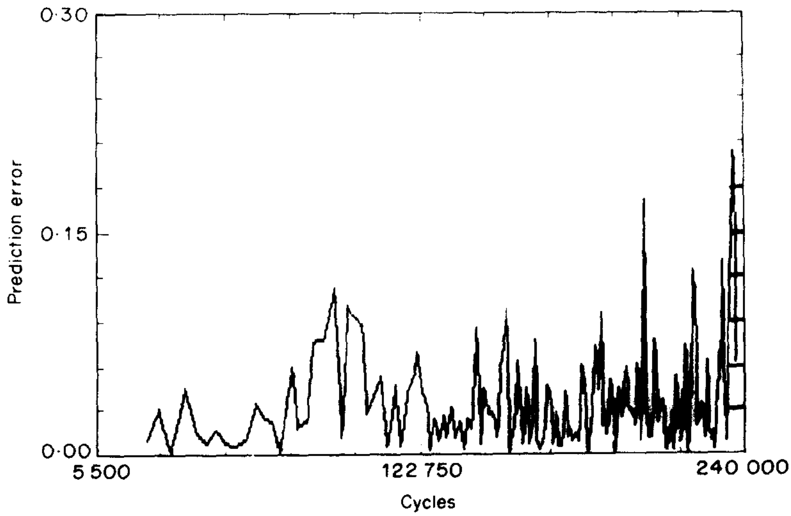


Figure 7. Absolute crack length prediction error for $n_W = 4$.

The failure condition, i.e. the condition that fast fracture will occur because of the defect, is at any number of cycles $N$,

$$a_c(q_1, q_2, \ldots, q_n) - a(q_1, q_2, \ldots, q_n) < 0.$$

In general, in order to take into account all sources of variability, the characterising parameters of the FCG phenomenon are random and time varying. This fact can be assessed very successfully, without the necessary mathematical complications arising from physical considerations [2-6, 10], using the linear regression procedure with moving window described in Section 4, to update the FCG law parameters after every inspection of the actual state of the structure. Moreover, the fast fracture phenomenon, the history dependence of FCG and the variability of the FCG physical phenomena can be well assessed by the same linear regression method.

In the FCG modeling method presented here, each specific case is related to the given geometry of an element and/or a crack, and should be treated separately. This means that the generality of the phenomenological FCG laws relating the crack growth with the stress intensity factor range is lost [1, 4, 5, 9, 10]. However, failures of structures occur in practice because of a few, and sometimes even only one, propagated defects considered the most dangerous. Thus, the generality of the method, when applied to prevent structural failure is not lost.

Moreover, the proposed moving window technique can be used to estimate the parameters of the most frequently used phenomenological FCG laws, as are the Paris-Erdogan and Forman's [9]. This can be accomplished by either using a large amount of laboratory replications or by updating after on-line inspections. It is well known that the curves of $\log_{10}(\mathrm{d}a/\mathrm{d}N)$ vs. $\log_{10}(\Delta K)$, where $\Delta K$ is the stress intensity factor range, are straight lines [1, 9]. Thus, estimation of the parameters $(c, m)$ of the Paris-Erdogan or Forman laws, which result in more general structural FCG predictions, can be performed using the recursive moving window method of Section 4. The recursive nature of the method makes it suitable for on-line inspection data treatment and thus it provides a unique tool for fast FCG predictions under any loading and temperature conditions. This is accomplished through the adaptive estimation of the parameters of the FCG model used and as a result all the occurring random conditions are taken into account irrespective of whether they are known or not.

To calculate the probability that fast fracture will occur because of the defect, equations (9), (10) and the approximate expressions given in Appendix A may be used, at any number of cycles $N$.

The whole computational procedure is very easy to implement in an existing computerised inspection and failure prediction scheme.

## 6. CONCLUSIONS

The basic problem in structural safety, that is to ensure that an inadmissible failure state will not occur during the lifetime of the structure, is treated in the present paper. An attempt to solve the generalised problem could be worthwhile. The state of the structure at any number of cycles, i.e. defect size and its critical value, would then be not only functions of the instantaneous thermomechanical loading but functionals depending on the entire loading history.

Linear regression techniques combined with FCG laws of exponential type with respect to the number of accumulated cycles, can be useful tools to overcome the difficulties of the structural integrity problem.

In particular, the moving window regression method presented here, can be easily incorporated in an existing safety monitoring system. In cases where crack length measurements are available on-line using appropriate hardware equipment, the recursive nature of the method makes it suitable for an integrated automatic safety alarm system.

The applicability of the moving window regression method is not limited to the simple FCG models adopted in this work, but can be easily extended to other well known phenomenological FCG laws for accurate and more general FCG predictions.

The failure probability of a structure is possible to be computed in a closed form, easy to evaluate at the engineering practice. Experimental FCG data were used to evaluate the proposed probability model for calculating the reliable life of structures in which crack propagation up to a critical length $a_c$ must be tolerated.

## REFERENCES


1. P. C. PARIS and F. ERDOGAN 1963 *Transactions of ASME, Journal of Basic Engineering* **85**, 528–534. A critical analysis of crack propagation laws.
2. J. L. BOGDANOFF and F. KOZIN 1985 *Probabilistic Models of Cumulative Damage*. New York: John Wiley.
3. F. KOZIN and J. L. BOGDANOFF 1985 *Probabilistic Methods in Mechanics of Solids and Structures*, S. Eggwertz and N. C. Lind, eds, pp. 93–101. Berlin: Springer-Verlag.
4. D. KUJAWSKI and F. ELLYIN 1984 *Engineering Fracture Mechanics* **20**, 695–704. A fatigue crack propagation model.
5. Y. K. LIN and J. N. YANG 1985 *AIAA Journal* **23**, 117–124. A stochastic theory of fatigue crack growth.
6. K. SOBCZYK 1986 *Engineering Fracture Mechanics* **24**, 609–623. Modelling of random fatigue crack growth.
7. A. LUCIA 1985 *Nuclear Engineering and Design* **87**, 35–49. Probabilistic structural reliability of PWR pressure vessels.
8. T. YOKOBORI and A. T. YOKOBORI 1985 *Proceedings of the 4th International Conference on Structural Safety and Reliability (ICOSSAR-85)*, pp. 319–329. Berlin: Springer-Verlag.
9. D. HOEPPNER and W. KRUPP 1974 *Engineering Fracture Mechanics* **6**, 47–70. Prediction of component life by application of fatigue crack growth knowledge.
10. R. ARONE 1986 *Engineering Fracture Mechanics* **24**, 223–232. Fatigue crack growth under random overloads superimposed on constant-amplitude cyclic loading.
11. F. R. SHANLEY 1956 *Colloquium on Fatigue, Stockholm, 1955*. A proposed mechanism of fatigue failure. Berlin: Springer Verlag.

12. W. CLARK and S. HUDAK 1975 *Journal of Testing and Evaluation* **3**, 454–464. Variability in fatigue crack growth rate testing.
13. D. VIRKLER, B. HILLBERRY and P. GOEL 1979 *ASME Journal of Engineering Materials Techniques 101.* The statistical nature of fatigue crack propagation.
14. R. BARLOW and F. PROSCHAN 1970 *Mathematical Theory of Reliability.* New York: John Wiley.
15. M. ABRAMOWITZ and I. STEGUN 1966 *Handbook of Mathematical Functions, Applied Mathematics Series No. 55.* Washington DC: National Bureau of Standards.

## APPENDIX A

Approximate expressions for fast engineering calculation of the structural reliability characterisations, when dealing with the normal distribution are as follows [15]:

$$\phi(s) = P(S \leq s) = 1 - \varphi(s)\left[\sum_{i=1}^{5} b_i t^i + \varepsilon(s)\right] \tag{A1}$$

where,

$$t = (1 + 0{\cdot}231\,6419s)^{-1}, \qquad |\varepsilon(s)| < 7{\cdot}5 \times 10^{-8}$$

$$b_1 = 0{\cdot}319\,381\,530, \qquad b_2 = -0{\cdot}356\,563\,782$$

$$b_3 = 1{\cdot}781\,477\,937, \qquad b_4 = -1{\cdot}821\,255\,978$$

$$b_5 = 1{\cdot}330\,274\,429$$

$$\phi(s) = P(S \leq s) = 1 - 0{\cdot}5\left(1 + \sum_{i=1}^{6} d_i s^i\right)^{-16} + \varepsilon(s) \tag{A2}$$

where,

$$|\varepsilon(s)| < 1{\cdot}5 \times 10^{-7}$$

$$d_1 = 0{\cdot}049\,867\,3470, \qquad d_2 = 0{\cdot}021\,141\,0061$$

$$d_3 = 0{\cdot}003\,277\,6263, \qquad d_4 = 3{\cdot}800\,36 \times 10^{-5}$$

$$d_5 = 4{\cdot}889\,06 \times 10^{-5}, \qquad d_6 = 0{\cdot}538\,30 \times 10^{-5}.$$

## APPENDIX B

Define,

$$\mathbf{U} = \begin{bmatrix} u_1(1) & \cdots & u_m(1) \\ \vdots & & \vdots \\ u_1(k) & \cdots & u_m(k) \end{bmatrix} = \begin{bmatrix} \mathbf{u}^T(1) \\ \vdots \\ \mathbf{u}^T(k) \end{bmatrix}$$

and,

$$\mathbf{y} = [y(1) \quad \cdots \quad y(k)]^T$$

Furthermore, for a moving window of length $n_W$, define,

$$\mathbf{U}_k = \begin{bmatrix} \mathbf{u}^T(k - n_W + 1) \\ \vdots \\ \mathbf{u}^T(k) \end{bmatrix} = \begin{bmatrix} \mathbf{u}^T(k - n_W + 1) \\ \hline \mathbf{U}(k, k - n_W + 2) \end{bmatrix}$$

Then,

$$\mathbf{U}_{k+1} = \begin{bmatrix} \mathbf{u}^T(k - n_W + 2) \\ \hline \vdots \\ \hline \mathbf{u}^T(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{U}(k, k - n_W + 2) \\ \hline \mathbf{u}^T(k+1) \end{bmatrix}.$$

The iteration for $\mathbf{U}_k^T\mathbf{U}_k$ is considered first.

$$\mathbf{U}_k^T\mathbf{U}_k = [\mathbf{u}(k-n_W+1)|\mathbf{U}^T(k, k-n_W+2)]\left[\begin{array}{c}\mathbf{u}^T(k-n_W+1)\\ \hline \mathbf{U}(k, k-n_W+2)\end{array}\right]$$

$$= \mathbf{u}(k-n_W+1)\mathbf{u}^T(k-n_W+1) + \mathbf{U}^T(k, k-n_W+2)\mathbf{U}(k, k-n_W+2)$$

and,

$$\mathbf{U}_{k+1}^T\mathbf{U}_{k+1} = [\mathbf{U}^T(k, k-n_W+2)|\mathbf{u}(k+1)]\left[\begin{array}{c}\mathbf{U}(k, k-n_W+2)\\ \hline \mathbf{u}^T(k+1)\end{array}\right]$$

$$= \mathbf{u}(k+1)\mathbf{u}^T(k+1) + \mathbf{U}^T(k, k-n_W+2)\mathbf{U}(k, k-n_W+2)$$

$$= \mathbf{U}_k^T\mathbf{U}_k + \mathbf{u}(k+1)\mathbf{u}^T(k+1) - \mathbf{u}(k-n_W+1)\mathbf{u}^T(k-n_W+1)$$

$$= \mathbf{U}_k^T\mathbf{U}_k + \mathbf{\Gamma}(k+1)$$

where,

$$\mathbf{\Gamma}(k+1) = \mathbf{u}(k+1)\mathbf{u}^T(k+1) - \mathbf{u}(k-n_W+1)\mathbf{u}^T(k-n_W+1)$$

Secondly, the iteration for $\mathbf{U}_k^T\mathbf{y}_k$ is considered. Define

$$\mathbf{y}_k = \begin{bmatrix} y(k-n_W+1)\\ \vdots \\ y(k)\end{bmatrix} = \left[\begin{array}{c} y(k-n_W+1)\\ \hline \mathbf{y}(k, k-n_W+2)\end{array}\right]$$

then,

$$\mathbf{y}_{k+1} = \begin{bmatrix} y(k-n_W+2)\\ \vdots \\ y(k+1)\end{bmatrix} = \left[\begin{array}{c} \mathbf{y}(k, k-n_W+2)\\ \hline y(k+1)\end{array}\right].$$

Hence

$$\mathbf{U}_k^T\mathbf{y}_k = [\mathbf{u}(k-n_W+1)|\mathbf{U}^T(k, k-n_W+2)]\left[\begin{array}{c} y(k-n_W+1)\\ \hline \mathbf{y}(k, k-n_W+2)\end{array}\right]$$

$$= \mathbf{u}(k-n_W+1)y(k-n_W+1) + \mathbf{U}^T(k, k-n_W+2)\mathbf{y}(k, k-n_W+2)$$

and,

$$\mathbf{U}_{k+1}^T\mathbf{y}_{k+1} = [\mathbf{U}^T(k, k-n_W+2)|\mathbf{u}(k+1)]\left[\begin{array}{c} \mathbf{y}(k, k-n_W+2)\\ \hline y(k+1)\end{array}\right]$$

$$= \mathbf{u}(k+1)y(k+1) + \mathbf{U}^T(k, k-n_W+2)\mathbf{y}(k, k-n_W+2)$$

$$= \mathbf{U}_k^T\mathbf{y}_k + \mathbf{u}(k+1)y(k+1) - \mathbf{u}(k-n_W+1)y(k-n_W+1)$$

$$= \mathbf{U}_k^T\mathbf{y}_k + \boldsymbol{\delta}(k+1)$$

where,

$$\boldsymbol{\delta}(k+1) = \mathbf{u}(k+1)y(k+1) - \mathbf{u}(k-n_W+1)y(k-n_W+1).$$

Now,

$$\hat{\boldsymbol{\theta}}(k+1) = (\mathbf{U}_{k+1}^T\mathbf{U}_{k+1})^{-1}\mathbf{U}_{k+1}^T\mathbf{y}_{k+1}$$

Defining

$$\mathbf{P}(k+1) = (\mathbf{U}_{k+1}^T \mathbf{U}_{k+1})^{-1}$$

which is the covariance of the estimate $\hat{\boldsymbol{\theta}}(k+1)$, we get

$$\begin{aligned}\hat{\boldsymbol{\theta}}(k+1) &= \mathbf{P}(k+1)[\mathbf{U}_k^T \mathbf{y}_k + \boldsymbol{\delta}(k+1)] \\ &= \mathbf{P}(k+1)[\mathbf{P}^{-1}(k)\hat{\boldsymbol{\theta}}(k) + \boldsymbol{\delta}(k+1)] \\ &= \mathbf{P}(k+1)[(\mathbf{P}^{-1}(k+1) - \boldsymbol{\Gamma}(k+1))\hat{\boldsymbol{\theta}}(k) + \boldsymbol{\delta}(k+1)] \\ &= \hat{\boldsymbol{\theta}}(k) - \mathbf{P}(k+1)[\boldsymbol{\Gamma}(k+1)\hat{\boldsymbol{\theta}}(k) - \boldsymbol{\delta}(k+1)] \qquad \text{(B1)}\end{aligned}$$

and,

$$\mathbf{P}^{-1}(k+1) = \mathbf{P}^{-1}(k) + \boldsymbol{\Gamma}(k+1). \qquad \text{(B2)}$$

Equations (B1) and (B2) form the moving window ordinary least squares estimator (MWOLS). Note that in this simple case a further reduction of (B2) is not needed since only one inversion is required.